Lecture XVIII. Separation theorems

<u>Theorem A)</u> $\forall$ 1-1 continuous map $f: D^q \to S^p$

$f: D^q \to S^p$, $\tilde{H}_n(S^p \setminus f(D^q)) = 0$ $(q \geq 0)$

B) $\forall$ 1-1 continuous map $f: S^q \to S^p$,

$$\tilde{H}_n(S^p \setminus f(S^q)) = \begin{cases} \mathbb{Z} & \text{if } n = p-q-1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Corollary</u> (Jordan-Brouwer sep. theorem)

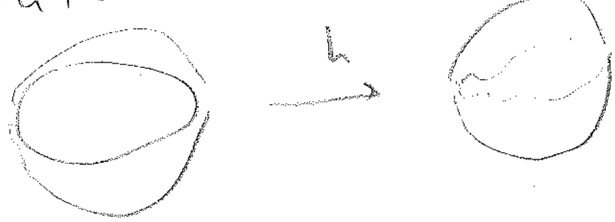
$\forall$ 1-1 continuous $f: S^{p-1} \to S^p$, $S^p \setminus f(S^{p-1})$ has exactly 2 components

<u>Proof</u> By B) $H_0(S^p \setminus f(S^{p-1})) \cong \mathbb{Z} \oplus \tilde{H}_0(S^p \setminus f(S^{p-1}))$
$\cong \mathbb{Z} \oplus \mathbb{Z}$

<u>Remark</u> for 1-1 cont $f: S^1 \to S^2$ more is true

(Schoenflies) $\exists$ homeomorphism $h: S^2 \to S^2$ s.t.

$f = h|\text{standard } S^1 \cup S^2$

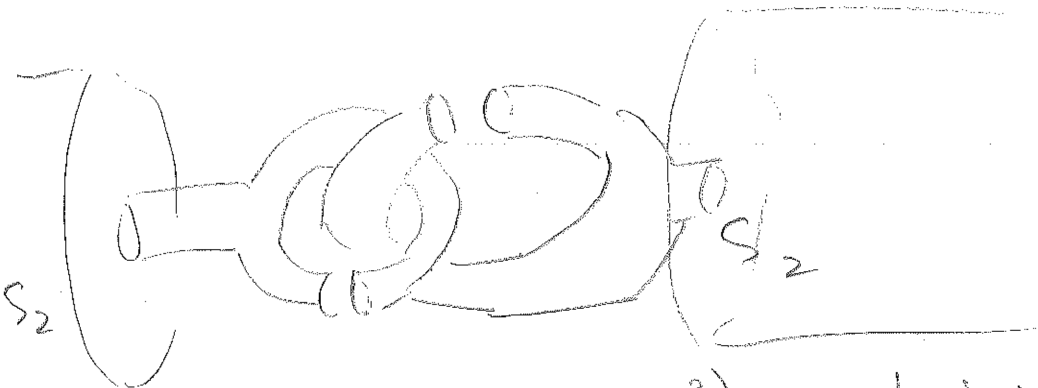


This fails in dim ≥ 3

That fails in dim ≥ 3

Alexander Horned Sphere

$S^2 \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$



$B^3 \overset{h}{\hookrightarrow} S^3$

$\pi_1(S^3 \setminus h(B^3))$ nontrivial

no hom. of $h(B^3)$ to a standard B-ball

Proof of A By induction on q, start at $q=0$

$D^q \cong D^{q-1} \times I \quad (q \geq 1)$

Let $z \in Z_n(S^p \setminus f(D^q))$

By induction $\forall\, t \in I$

$\exists\, c \in C_{n+1}(S^p \setminus f(D^{q-1} \times \{t\}))$ s.t. $\partial c_t = z$

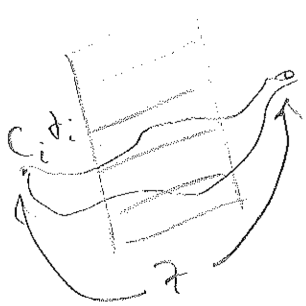
c_t is a finite combination of singular

$n+1$-simplices

Compactness: $\exists\, \varepsilon > 0$ s.t. $c_t \in C_{n+1}(S^p \setminus f(D^{q-1} \times (t-\varepsilon, t+\varepsilon)))$

$\exists$ dissection $0 = t_0 < t_1 < \dots < t_k = 1$ s.t.

$z = \partial c_i$ for some $c_i \in C_{n+1}(S^p \setminus f(D^{q-1} \times [t_{i-1}, t_i]))$



Suppose (induction) that z repr.

0 in $H_n(S^p \setminus f(D^{q-1} \times [0, t_i]))$

Mayer-Vietoris: $S^p \setminus f(D^{q-1} \times [0, t_i]) \cap S^p \setminus f(D^{q-1} \times [t_i, t_{i+1}])$

$= S^p \setminus f(D^{q-1} \times t_i)$

$H_{n+1}(S^p \setminus f(D^{q-1} \times t_i)) \to \underset{[z]}{H_n(S^p \setminus f(D^{q-1} \times [0, t_{i+1}]))} \to$ (0 by ind. on q)

$\to H_n(S^p \setminus f(D^{q-1} \times [0, t_i])) \oplus H_n(S^p \setminus f(D^{q-1} \times [t_i, t_{i+1}]))$

$(0 \qquad\qquad 0)$

$\Rightarrow z$ repr. 0 in $H_n(S^p \setminus f(D^{q-1} \times [0, t_{i+1}]))$ in $H_n(S^p \setminus f(D^{q-1} \times I))$

Proof B)

$S^q = D^q_+ \cup D^q_-$, $D^q_+ \cap D^q_- = S^{q-1}$

$S^p \setminus f(D^q_+) \cap S^p \setminus f(D^q_-) =$

$= S^p \setminus f(S^{q-1})$

$S^p \setminus f(D^q_+) \cup S^p \setminus f(D^q_-) = S^p \setminus f(S^q)$

Mayer-Vietoris

$H_{n+1}(S^p \setminus f(D^q_+)) \oplus H_{n+1}(S^p \setminus f(D^q_-)) \to$

$\to H_{+1}(S^p \setminus f(S^{q-1})) \to H_n(S^p \setminus f(S^q)) \to$

$\to \underbrace{\tilde{H}_n(S^p \setminus f(D^q_+)) \oplus \tilde{H}_n(S^p \setminus f(D^q_-))}_{0}$

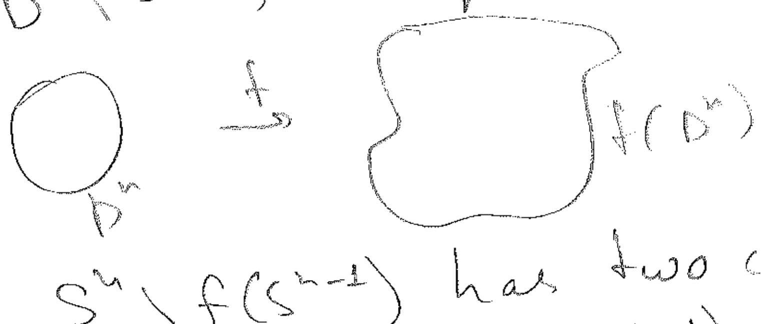
$\tilde{H}_n(S^p \setminus f(S^q)) \cong \tilde{H}_{n+1}(S^p \setminus f(S^{q-1})) \cong \dots$

$\cong \tilde{H}_{n+q}(S^p \setminus f(S^0)) \cong \tilde{H}_{n+q}(S^{p-1}) \cong \begin{cases} \mathbb{Z} & \text{if } n = p-q-1 \\ 0 & \text{otherwise} \end{cases}$

Corollary For any 1-1 continuous $f : D^n \to S^n$

$f(D^n \setminus S^{n-1})$ is open in S^n

$S^n = \mathbb{R}^n \cup \{\infty\}$



Proof $S^n \setminus f(S^{n-1})$ has two components:

$S^n \setminus f(S^{n-1}) = f(D^n \setminus S^{n-1}) \cup (S^n \setminus f(D^n))$

$\Rightarrow f(D^n \setminus S^{n-1})$ is a comp of $S^n \setminus f(S^{n-1})$

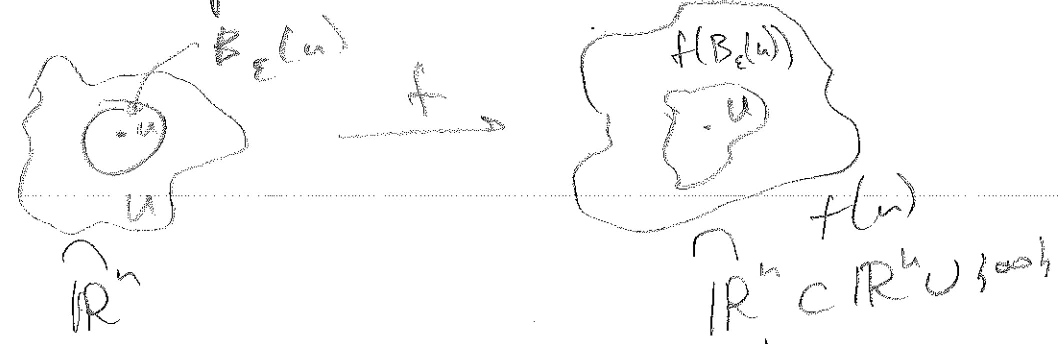
$f(D^n \setminus S^{n-1})$ - open

Corollary (Invariance of domain)

If $\mathcal{U}$ is open in $\mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ 1-1 continuous

$\Rightarrow f(\mathcal{U})$ is open in $\mathbb{R}^n$

Proof.



$u \in \mathcal{U}$ $\exists \varepsilon > 0$ s.t. $\mathcal{U} \supset B_\varepsilon(u) \cong D^n$

$f(B_\varepsilon(u) \setminus \partial B_\varepsilon(u))$ open in $\mathbb{R}^n \cup \{\infty\}$ contained in $f(\mathcal{U})$