Lecture XIX
Extra techniques for homotopy groups

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The road so far:
We defined $\pi_n(X, A, x_0)$ and showed that there is the exact sequence, involving $\pi_n(X, A, x_0)$:

$$\to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

One can write a similar for the triple:

$$\to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \to \pi_{n-1}(A, B, x_0) \to \pi_1(X, A, x_0)$$

Whitehead theorem:

If a map $X \to Y$ is between conn. CW complexes induces isomorphism $f_*: \pi_n(X) \to \pi_n(Y)$ $\forall n$ then f is a homotopy eq. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X - def. retract of Y

Follows from Compression lemma:
Let (X, A) be a CW pair and let (Y, B) be the pair $B \neq \emptyset$. For each n such that $X \setminus A$ has cells of dim $= n$, assume that $\pi_n(Y, B, y_0) = 0$ $\forall y_0 \in B$. Then every map $f: (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$

Lemma (extension) Given a CW pair (X, A) and a map $f: A \to Y$ with Y path-conn., then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ $\forall n$ s.t. $X \setminus A$ has cells of dim n

Proof Assume inductively that f has been extended over the $(n-1)$-skeleton. Then an extension over an n-cell exists iff the composition of cell's att. map $S^{n-1} \to X^{n-1}$ with $f: X^{n-1} \to Y$ is null-hom.

Theorem $\forall f: X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, the homotopy may be taken to be stationary on A.

Corollary $\pi_n(S^k) = 0$ for $n < k$

Elementary methods of computation

Theorem Let X - CW complex, A, B s.t. $A \cup B = X$, $A \cap B = C$ = connected, nonempty. If (A, C) is m-conn. and (B, C) - n-connected; $m, n \geq 0$, then $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isom. for $i < m+n$ and surjection for $i = m+n$

Corollary $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isom. for $i < 2n-1$ and a surjection for $i = 2n-1$.
More generally: $\pi_i(X) \to \pi_{i+1}(SX)$ where X is an $(n-1)$-connected CW complex.

Proof: Decompose $SX = C_+X \cup C_-X$ $\quad C_+X \cap C_-X = X$

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \xrightarrow{\text{inclusion}} \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX)$$

long exact sequences

From long exact seq. of (C_+X, X) we see that it is n-conn. if X is $(n-1)$-conn. $\Rightarrow$ preceding theorem says middle map is an isom for $i+1 < 2n$ and surj for $i+1 = 2n$.

Corollary $\pi_n(S^n) \cong \mathbb{Z}$, generated by the id. map for all $n \geq 1$. In particular the deg. map $\pi_n(S^n) \to \mathbb{Z}$ is an isom. but we proved from Hopf bundle.

$$\pi_1(S^1) \underset{\text{surj.}}{\longrightarrow} \pi_2(S^2) \underset{\text{isom}}{\longrightarrow} \pi_3(S^3) \longrightarrow$$

More on computation

Proposition If (X,A) - CW pair which is r-connected, A - s-connected with $r,s \geq 0 \Rightarrow$ map $\pi_i(X,A) \to \pi_i(X/A)$ induced by $X \to X/A$ is an isom. for $i \leq r+s$ and a surj for $i = r+s+1$

Proof Consider $X \cup CA$.

CA - contractible subcomplex of $X \cup CA$

then $X \cup CA \to (X \cup CA)/CA = X/A$ - homotopy eq.

$$\begin{tikzcd} \pi_i(X,A) \arrow[r] & \pi_i(X \cup CA, CA) \arrow[r] & \pi_i(X \cup CA/CA) = \pi_i(X/A) \\ & \pi_i(X \cup CA) \arrow[u, "\cong"] \arrow[ur, "\cong"] & \end{tikzcd}$$

apply excision th. using fact that (CA, A) - $(s+1)$-conn. if A is s-conn.

long exact seq.

Example X obtained from $\vee_\alpha S^n_\alpha$ by attaching e^{n+1}_β via $\varphi_\beta : S^n \to \vee_\alpha S^n_\alpha$ with $n \geq 2$

Cellular approx: $\pi_i(X) = 0$ for $i < n$

Now show that $\pi_n(X)$ is the quotient of $\pi_n(\vee_\alpha S^n_\alpha) \cong \bigoplus_\alpha \mathbb{Z}$ by $[\varphi_\beta]$

Let us return back to that example:

$\pi_n(S^n) \to \pi_n(\vee_\alpha S^n_\alpha)$ $n \geq 2$ is free abelian with basis the hom. classes of $S^n_\alpha \to \vee_\alpha S^n_\alpha$

Suppose that it is a finite wedge.

Pair $(\prod_\alpha S^n_\alpha, \vee_\alpha S^n_\alpha)$ - $2n-1$ connected $\Rightarrow$

$\Rightarrow \vee_\alpha S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$ induces isom on π_n if $n \geq 2$

$\Rightarrow \pi_n(\prod_\alpha S^n_\alpha) \cong \bigoplus_\alpha \pi_n(S^n_\alpha)$ $\Rightarrow$ same is true for $\vee_\alpha S^n_\alpha$

<u>Ex.</u> Finish: prove inf. dim case.

To see that π_n as claimed

$$\pi_{n+1}(X, \vee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\vee_\alpha S^n_\alpha) \to \pi_n(X) \to 0$$

Quotient $X / \vee_\alpha S^n_\alpha$ is a wedge of spheres S^{n+1}_β.

Prev. prop. then implies that $\pi_{n+1}(X, \vee_\alpha S^n_\alpha)$ is free with basis the char. cells of e^{n+1}_β

∂ takes them to $[\varphi_\beta]$.

<u>Eilenberg-MacLane spaces</u>

X with 1 nontrivial hom. group $\pi_n(X) \cong G$ is called E-M space $K(G,n)$

<u>Construction</u> $\mathbb{CP}^\infty$ $K(\mathbb{Z},2)$ - most. example

X — $(n-1)$-connected CW complex of dim $n+1$, s.t. $\pi_n(X)\cong G$ as in ex. above and apply Postnikov tower

<u>Proposition</u> Homotopy type of a CW complex $K(G,n)$ is uniquely det. by G and n.

<u>Lemma</u> Let X be a CW complex $(\vee_\alpha S^n_\alpha)\cup_\beta e^{n+1}_\beta$ for some $n\geq 1$. Then for every hom. $\varphi:\pi_n(X)\to\pi_n(Y)$ with Y path connected $\exists$ map $f:X\to Y$ with $f_*=\varphi$.

Was in the hmwk.

<u>Proof of Prop</u>

Suppose K, K' are $K(G,n)$ CW complexes.

Since homotopy equivalence is eq. rel, assume $K=X$ constructed from $\vee S^n_\alpha$. By lemma $f:X\to K'$ from isom induce isom on homotopy groups

<u>Postnikov tower</u>

X conn. CW complex X_n, containing X as subcomplex

and a) $\pi_i(X_n)=0$ for $i>n$

b) $X\hookrightarrow X_n$ induces isom π_i for $i\leq n$

attach $n+2$ cells to X using cellular maps $S^{n+1}\to X$ that generate $\pi_{n+1}(X)$ etc.

$$\begin{tikzcd} & X_3 \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & X_1 \end{tikzcd}$$

Hurewicz theorem

Theorem If a space X is $(n-1)$-connected, $n \geq 2$ then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. If a pair (X, A) is $(n-1)$-connected, $n \geq 2$, with A simply-conn. and nonempty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$

Proof

Extra result 1) CW-approx:
$\forall$ top. space X $\exists$ CW complex Z and a weak homotopy eq. $f: Z \to X$

We assume that (X, A) – CW pair

For CW pair rel case follows from absolute b/c $\pi_i(X, A) \cong \pi_i(X/A)$ for $i \leq n$ and $H_i(X, A) \cong \tilde{H}_i(X/A)$

Extra result 2) If (X, A) is an n-conn. CW pair $\Rightarrow$ $\exists$ CW pair $(Z, A) \simeq (X, A)$ rel A s.t. all cells of $Z - A$ have dim greater than n

Replace X with hom. eq. CW complex with $(n-1)$-skeleton being a pt. $\Rightarrow \tilde{H}_i(X) = 0$ for $< n$

To show $\pi_n(X) \cong H_n(X)$, throw away cells $\geq n+2$

X has the form: $\left(\bigvee_\alpha S^n_\alpha\right) \cup_\beta e^{n+1}_\beta$

$$\underset{\oplus_\beta \mathbb{Z}}{\pi_{n+1}(X, X^n)} \longrightarrow \underset{\oplus_\alpha \mathbb{Z}}{\pi_n(X^n)}$$ cokernel of this map.

Same as: $d: H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$

<u>Corollary</u> A map $f: X \to Y$ between simply conn. CW complexes is homotopy eq. if $f_*: H_n(X) \to H_n(Y)$ is isom for each n

<u>Proof</u> Replace Y by mapping cylinder M_f

$M_f = (([0,1] \times X) \sqcup (Y))/\sim \quad (0,x) \sim f(x)$

so $X \underset{\text{inclusion}}{\hookrightarrow} Y \quad \pi_1(Y,X) = 0 \Rightarrow$ first nonzero $\pi_n(Y,X)$ is isom to $H_n(Y,X)$. All $H_n(Y,X)$ are zero from long exact seq $\Rightarrow \pi_n(Y,X)$ also vanish $\Rightarrow$ isom.