# Quantum Field Theory

Two approaches $\nearrow$ operator, $\searrow$ path integral approach $\leftrightarrow$ Geometry

"Shadow" QFT : TQFT (n-dim)

1) $n-1$-dim oriented compact manifold $\Sigma \to$ finite-dim complex vectorspace $\mathcal{H}_\Sigma$ "space of states"

2) $n$-dim. compact oriented manifold $M$ with boundary $\to$ vector $Z_M \in \mathcal{H}_\Sigma$

A1 Involutivity $\mathcal{H}_{\Sigma^*} = \mathcal{H}_\Sigma^*$

A2 Multiplicativity $\mathcal{H}_{\Sigma_1 \cup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$

Remark. $\partial M = \Sigma_{out} \cup \Sigma_{in}^*$

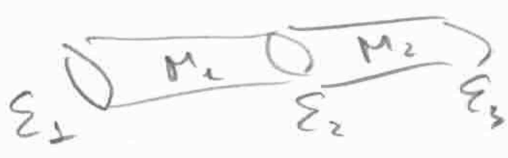
Operator: $Z_M : \mathcal{H}_{\Sigma_{in}} \to \mathcal{H}_{out}$

Associativity: $Z_M = Z_{M_2} Z_{M_1}$

$M = M_1 \cup_{\Sigma_2} M_2$

$\partial M_1 = \Sigma_2 \cup \Sigma_1^*$

$\partial M_2 = \Sigma_3 \cup \Sigma_2^*$




A4 $\mathcal{H}_\emptyset = \mathbb{C}$ $\quad \emptyset$ – empty $n-1$-manifold

Functoriality: $\phi : M \to M'$ $\in \mathrm{Diff}^+$

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_{in}} & \xrightarrow{Z_M} & \mathcal{H}_{\Sigma_{out}} \\ \downarrow {\scriptstyle \rho(\phi|_{in})} & & \downarrow {\scriptstyle \rho(\phi_{out})} \\ \mathcal{H}_{\Sigma'_{in}} & \xrightarrow{Z_{M'}} & \mathcal{H}_{\Sigma'_{out}} \end{array}$$

Unitarity: $Z_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$

$Z_{M^*} : \mathcal{H}_{\Sigma_2} \to \mathcal{H}_{\Sigma_1}$

conjugate

A5 $Z_{\Sigma \times I} = id_{\mathcal{H}_\Sigma}$

Corollaries

1) Take $\Sigma_1 = \Sigma_3 = \phi$, $\Sigma_2 = \Sigma$

$$Z_M = \langle Z_{M_1}, Z_{M_2} \rangle \in \mathbb{C}$$

pairing


One can take any $\Sigma$-cut: $M = M_1 \cup_\Sigma M_2$

2) Functoriality + AS

$\Downarrow$

$\rho(\phi)$ - depends only on mapping class group of $\Sigma$

3) Construct $\Sigma_\phi$ by identifying boundary of cylinder $\Sigma \times [0,1]$ via $\phi \in \mathrm{Diff}^+(\Sigma)$

$$\mathrm{Tr}\, \rho(\phi) = Z_{\Sigma_\phi}$$

In particular,

$$\dim \mathcal{H}_\Sigma = Z_{\Sigma \times S^1}$$

Ex. 1) $n = 1$. Everything is determined by a pt: $\mathcal{H}_{pt} = \mathbb{C}^N$ $Z_{S^1} = N$

2) $n = 2$ (Dijkgraaf)

Frobenius algebra.

$(\alpha \cdot \beta, \gamma) = (\alpha, \gamma \cdot \beta)$

$\mathcal{H}_{S^1} = V$

$Z$  $m : V \otimes V \to V$

$Z$  unit $1 \in V$ $1 : \mathbb{C} \to V$

$Z$  counit $\epsilon : V \to \mathbb{C}$

$Z$  pairing $(\,,\,) : V \otimes V \to \mathbb{C}$

$Z$  coproduct $\Delta : V \to V \otimes V$

$$Z_{\text{closed surface of genus } g} = \epsilon \circ (m \circ \Delta)^g \, 1$$

3) $n \geq 3$ Hard. Needed a refinement

Ex. Cohomology, quantum cohomology of compl. $X$

$\mathbb{P}^N$ $x^{N+1} = 0 \to q$ $\alpha \cdot \beta = \alpha \wedge \beta$ $\int_X \alpha \wedge \beta = \eta(\alpha, \beta)$

# Segal's QFT

Manifold with some geometric data: Riemann metric, complex structure, etc

Modification of Atiyah axioms to include geom. data:

Gluing: $Geom_M = Geom_{M_2} \times_{Geom_{\Sigma_2}} Geom_{M_1}$

functoriality:

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_{in}, \gamma_{in}} & \xrightarrow{Z_{M,\gamma}} & \mathcal{H}_{\Sigma_{out}, \gamma_{out}} \\ \downarrow S(\phi|_{in}) & & \downarrow S(\phi|_{out}) \\ \mathcal{H}_{\Sigma'_{in}, \phi_* \gamma_{in}} & \xrightarrow{Z_{M', \phi_* \gamma}} & \mathcal{H}_{\Sigma'_{out}, \phi_* \gamma_{out}} \end{array}$$

$\phi_*$ - pushforward.

## Ex. Quantum mechanics

$\pm$ / orientation for pts:
$+ \to \mathcal{H}$
$- \to \mathcal{H}^*$

Metric on 1-dim. cobordisms

$Z(t) = Z_{I_t} \in End(\mathcal{H})$

$I_t$ – interval of length $t$

$Z(t_1 + t_2) = Z(t_1) Z(t_2)$

Assume theory is unitary:

Unitarity: $Z_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$, $Z_{M^*} : \mathcal{H}_{\Sigma_2} \to \mathcal{H}_{\Sigma_1}$ } conjugate

$Z_M^*(t) = Z_{\bar{M}}(t)$ for cylinder

$Z(t) = e^{-\frac{i}{\hbar} H t}$ ← Hamiltonian.

$\mathcal{H} = L^2(X)$ $\quad \psi(x) \to \int_{x \in y} dy\, Z(t, x, y) \psi(y)$ ← operator.

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## Path integrals

Classical field theory on $n$-manifold

i) Space of fields:

$F_M = \Gamma(M, F_M)$ - sections of some "sheaf"

1) functions
2) sections of vector bundles
3) Maps: $M \to N$ ← target space "sigma-models"
4) Connections on principal $G$-bundles over $M$
   Chern-Simons Yang-Mills

ii) Action functional $S_M : F_M \to \mathbb{R}$

$S_M(\phi) = \int_M \mathcal{L}(\phi)$ - lagrangian density

Ex. Quantum mechanics in $\mathbb{R}^N$

$$S_M(\gamma) = \int_{t_1}^{t_2} \Big(\sum_i \frac{m\dot{q}_i^2}{2} - V(q)\Big)\, dt$$

potential

$\gamma : \mathbb{R} \to \mathbb{R}^N$

$t \to q_i(t)$

Path integral quantization:

$$Z_M(\hbar) = \int_{F_M} D\phi\, e^{\frac{i}{\hbar} S_M(\phi)}$$

Hard to define. Easy to consider asymptotic expansion $\hbar \to 0$.

Take M with boundary $B_\Sigma$ - boundary values of fields.

Take pullback $\Sigma \xhookrightarrow{i} M$ $\quad \phi \xrightarrow{i^*} \phi|_\Sigma$

$$\mathcal{H}_\Sigma = \mathrm{Fun}_{\mathbb{C}}(B_\Sigma)$$

$$Z_M(\phi_\Sigma, \hbar) = \int_{\phi \in F_M,\ \phi|_{\partial M} = \phi_\Sigma} D\phi\, e^{\frac{i}{\hbar} S_M(\phi)}$$

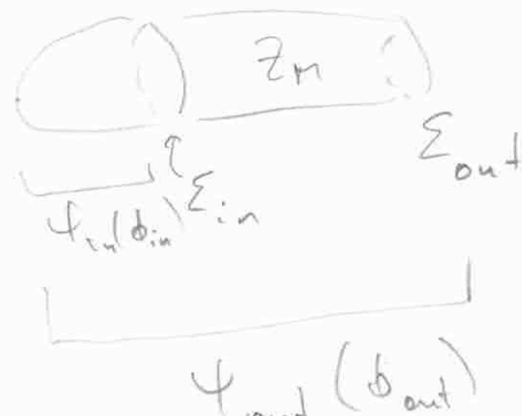
Argument for gluing:

$$Z_M = \int D\phi_\Sigma\, Z_{M_1}(\phi_\Sigma)\, Z_{M_2}(\phi_\Sigma)$$

$B_\Sigma \ni \phi_\Sigma$


$M = M_1 \cup_\Sigma M_2$

$$\psi_{out}(\phi_{out}) = \int D\phi_{in}\, Z_M(\phi_{out}, \phi_{in})\, \psi_{in}(\phi_{in})$$



<u>Path integral - making sense</u>

$f \in C^\infty(X) \quad \int_X \mu \, e^{\frac{i}{\hbar} f(x)}$

Stationary phase formula:

$$\int_X \mu \, e^{\frac{i}{\hbar} f(x)} \underset{\hbar \to 0}{\sim} \sum_{x_0 \in \text{crit. points of } f} e^{\frac{i}{\hbar} f(x_0)} |\det f''(x_0)|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}} + o(\hbar^{n/2})$$

positive - negative eigenv.

$df|_{x_0} = 0$ - "stationary phase points" oscillations slow down

Improvement:

$$\int_X \mu \, e^{\frac{i}{\hbar} f(x)} \underset{\hbar \to 0}{\sim} \sum_{x \in \text{crit. pt. of } f} e^{\frac{i}{\hbar} f(x_0)} |\det f''(x_0)|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Phi_\Gamma$$

vertices of valency $\geq 3$

$-\chi(\Gamma) = \#E - \#V$

Feynman diagrams

$\chi(\Gamma) \leq 0$

euler char. of graph

half-edges - decorated w indices

i) Every edge:

$\to f''(x_0)^{-1}_{i_1, i_2}$ - inverse of Hessian

ii) vertex:

$\partial_{i_1} \ldots \partial_{i_k} f(x_0)$ - k-th partial derivative at crit. pt.

$$\Phi_\Gamma = i^{E+V} \sum_{i_1, \ldots i_{2E} \in \{1, \ldots n\}} \prod_{\text{edges } e=(h_1,h_2)} f''(x_0)^{-1}_{i_1, i_2} \prod_{\text{vertices } v} \partial_{i_1} \ldots \partial_{i_{val(v)}} f(x_0)$$

<u>Ex.</u>

Θ-graph

$\chi(\Theta) = -1$

$\mathbb{Z}_2 \times S_3$

symmetry group

$\frac{\hbar}{12}$ - coeff.

$\Phi\Big($ 

$\Big) = i^{3+2} \ast$

$$\sum_{i,j,k,l,m,n} f''(x_0)^{-1}_{il} \, f''(x_0)^{-1}_{jm} \, f''(x_0)^{-1}_{kn} \, f'''(x_0)_{ijk} \, f'''(x_0)_{lmn}$$

Problem: "Gauge symmetry"

Instead of isolated fixed pts
There is a continuous symmetry
e.g. gauge group $G$. How to isolate $G$-orbits?

BV-formalism:

$\{F, S\} \longrightarrow$ new data:

i) $\mathcal{F}$ graded supermanifold $\mathcal{F}$
"space of BV-fields"
$\omega$ - odd symplectic structure of deg $=1$

ii) $S_{BV}$ on $\mathcal{F}$:

$\{S_{BV}, S_{BV}\} = 0$ deg 1 Poisson bracket

$\{S_{BV}, \cdot\} = Q$

$Q^2 = 0 \Rightarrow \mathcal{F}$ - dg manifold

$$\int_F e^{\frac{i}{\hbar} S} \longrightarrow \int_{\mathcal{L} \subset \mathcal{F}} e^{i/\hbar \, S_{BV}}$$

Lagrangian submanifold.

Relative TQFT:

Example: $M$ - 3d manifold

$L \subset M$ - oriented 1-dim submanifold

$\partial M = \Sigma$ assume $L$ transversal to $\Sigma$ and $+$ $-$ pts

Indices

Jones-Witten CS    repres of $G$