Connections on vector bundles

X - v.field on $U \subset \mathbb{R}^m$

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \qquad v \in T_q U = \mathbb{R}^m$$

$$(D_v X)_q = \sum_i (D_v a_i) \frac{\partial}{\partial x_i}\Big|_q$$

directional derivative of a function

One can say that vector field does not change along γ if $D_{\dot\gamma} X = 0$

What about manifolds?

Def. $\pi : E \to M$ - vector bundle.

$$\nabla^A : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$
$$(X, s) \to \nabla^A_X s$$

1) $\nabla^A_{fX} s = f \nabla_X s$

2) $\nabla^A_X (fs) = X(f) \cdot s + f \nabla^A_X s$

$f \in C^\infty(M)$, $X \in \Gamma(TM)$, $s \in \Gamma(E)$

Ex. Trivial vector bundle:

$s_1, \dots, s_e$ - frame i.e. $\{s_i(x)\}$-basis global sections

$$\nabla^A_X s = \nabla^A_X \left(\sum_i f_i s_i\right) = \sum_i X(f_i) s_i$$

Def. Generalized Christoffel symbols:

$(x_1, \dots, x_n) : U \to \mathbb{R}^n$ coord. chart

$\{e_\alpha\}$ frame on $E|_U$ i.e.

any section $\sigma = \sum \sigma^\alpha e_\alpha$

$$\nabla^A_{\frac{\partial}{\partial x_i}} e_\alpha = \sum_\alpha \omega_i{}^\beta{}_\alpha \, e_\beta$$

C.S. of connection ∇ relative to coord. $(x_1, \dots, x_n)$ and frame $\{e_\alpha\}$

$$\boxed{\nabla^A_X \sigma} = \nabla^A_{\sum x^i \frac{\partial}{\partial x_i}} \left(\sum \sigma^\alpha e_\alpha\right) =$$

$$= \sum x^i \frac{\partial \sigma^\alpha}{\partial x_i} e_\alpha + \sum x^i \sigma^\alpha \nabla_{\frac{\partial}{\partial x_i}} e_\alpha =$$

$$= \boxed{\sum X^i \left(\sum \frac{\partial \sigma^\alpha}{\partial x_i} e_\alpha + \sum \sigma^\beta \omega_i{}^\alpha{}_\beta \, e_\alpha\right)}$$

On the other hand, connection is:

Def. $d_A : \Gamma(E) \to \Gamma(\Omega^1(M) \otimes E)$

$$d_A(f\sigma + \tau) = df \otimes \sigma + f d_A \sigma + d_A \tau$$

Equivalence: $d_A(\sigma)(X) = \nabla^A_X \sigma$

$$d_A \sigma = d\sigma + \omega \sigma$$

matrix of 1-forms.

$$d_A \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^m \end{pmatrix} = \begin{pmatrix} d\sigma^1 \\ \vdots \\ d\sigma^m \end{pmatrix} + \begin{pmatrix} \omega^1_1 & \dots & \omega^1_m \\ \vdots & & \\ \omega^m_1 & \dots & \omega^m_m \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \vdots \\ \sigma^m \end{pmatrix}$$

vector bundle transition f-ns.

$\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^m$

$s: M \to E \quad \pi \circ s = id$

$s_U = g_{UV} s_V$

transition functions

$\varphi_U \circ \varphi_V^{-1}(x, v) = (x, g_{UV}(x) v)$ - on overlap

Cocycle: $g_{UU} = Id$. $g_{UV} g_{VW} g_{WU} = Id$

$(d_A s)_U = d s_U + \omega_U s_U$

$d s_U + \omega_U s_U = g_{UV} (d s_V + \omega_V s_V) =$ on $U \cap V$

$= g_{UV}(d(g_{UV}^{-1} s_V) + \omega_V g_{UV}^{-1} s_U) =$

$= d s_V + (g_{UV} d g_{UV}^{-1} + g_{UV} \omega_V g_{UV}^{-1}) s_U$

$\Rightarrow \omega_U = g_{UV} d g_{UV}^{-1} + g_{UV} \omega_V g_{UV}^{-1}$

Def 3: Connection on a vector bundle E defined by $(\{U\}, \{g_{UV}\})$ is a collection of $\{\omega_U\}$ with transf.

$\omega_U = g_{UV} d g_{UV}^{-1} + g_{UV} \omega_V g_{UV}^{-1}$

Parallel transport:

Pull-back bundle with a connection:

$F: N \to M$ - smooth map

$(F^* E, \pi^*)$ on N:

consider coverings $F^{-1}(U)$, transition functions $g_{UV} \circ F$

$F^* E = \{(p, v) \in N \times E : F(p) = \pi(v)\}$

$\pi^*((p, v)) = p$

Pull-back connection:

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{d_A} & \Gamma(\Omega^1(M) \otimes E) \\ \downarrow F^* & & \downarrow F^* \\ \Gamma(F^*E) & \xrightarrow{d_{F^*A}} & \Gamma(\Omega^1(M) \otimes E) \end{array}$$

$\gamma : [a,b] \to M$ - smooth curve

$(\gamma^* E, \pi^*)$ - bundle over $[a, b]$

$\gamma^* E$ - Trivial bundle : $[a,b] \times \mathbb{R}^m$

$d_{\gamma^* A} = d + \omega \qquad \omega = \{f^\alpha_\beta\} dt$

Par transp. $d_{\gamma^* A} s = ds + \omega s = 0$

$$\frac{d s^\alpha}{dt} + \omega^\alpha_{i\beta}(x) \frac{dx^i}{dt} s^\beta = 0$$

$\tau : (\gamma^* E)_a \to (\gamma^* E)_b$
isomorphism along the flow
Since $(\gamma^* E)_a \cong E_a$ $(\gamma^* E)_b \cong E_b$

Def: $\tau : E_{\gamma(a)} \to E_{\gamma(b)}$ is called parallel transport

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Remark $\nabla_X \, s = 0$ is the eq. of parallel transp.

$X^i = \frac{dx^i}{dt}$
tangent vector.

Def ∇ - connection on $TM \to M$
$\gamma : [a,b] \to M$ is geodesic if
$\dot{\gamma}(t)$ is parallel: $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$

$$\omega_i{}^j{}_k(x) = \Gamma^j_{ik}(x)$$

$$\frac{d^2 x^j}{dt^2} + \Gamma^j_{ik} \frac{dx^i}{dt} \frac{dx^k}{dt} = 0$$

Ex. On $\mathbb{R}^n$ $\Gamma^j_{ik} = 0$

$$\frac{d^2 x^i}{dt^2} = 0$$ - stright lines.

Curvature of the connection

1) $\Omega^p(E) = \Gamma(\Omega^p(M) \otimes E)$

$d_A : \Omega^0(E) \to \Omega^1(E)$

$d_A(\rho \otimes s) = d\rho \otimes s + (-1)^p \rho \wedge d_A s$

$\omega \in \Omega^p(M)$, $s \in \Gamma(E)$
can be extende d to any $\Omega^p(E)$

2) Similarly ∇^A_X can be extended to tensors of $\Gamma(T^{r,s}(E))$

$\nabla^A_X s^*$: $\nabla^A_X \langle s^*, s\rangle = d\langle s^*, s\rangle$
$= \langle \nabla^A_X s^*, s\rangle +$
$+ \langle s^*, \nabla^A_X s\rangle$

$\nabla^A_X \, s_1 \otimes s_2 = \nabla^A_X s_1 \otimes s_2 +$
$+ s_1 \otimes \nabla^A_X s_2$

$d_A^2 \overset{?}{=} 0$

$(d_A \circ d_A s) = d_A(ds + \omega s)$
$= d(ds) + d\omega s - \omega \wedge ds +$
$+ \omega \wedge ds + \omega \wedge \omega s = \Omega s$

$\boxed{3\tfrac{1}{2}}$

Summary:

Connection A on E:

1) $d_A : \Omega^p(E) \to \Omega^{p+1}(E)$

$$d_A s|_u = ds|_u + \omega_u \wedge s|_u$$

$$\{\omega^{\alpha}_{i\beta} dx^i\} = \omega_u \in \Omega^1_u(\mathrm{End}(E))$$

2) $\nabla_X : T^{r,s}(E) \to T^{r,s}(E)$

$X \in \Gamma(TM)$

On $\Gamma(E)$ $\quad \nabla_X s = d_A s(X)$

3) $\omega_u = g_{uv}\, d g_{uv}^{-1} + g_{uv}\, \omega_v\, g_{uv}^{-1}$

Gauge transf: $h_u^{-1} \tilde{g}_{uv} h_v = g_{uv}$

equivalent bundle

$$\tilde{\omega}_v = h_v d h_v^{-1} + h_v \omega_v h_v^{-1}$$

Parallel transport along the curve:

$$\frac{d z^{\alpha}(x)}{dt} + \sum_{i,\beta} \omega^{\alpha}_{i\beta}(x) \frac{dx^i}{dt} z^{\beta}(x) = 0$$

$\nabla_{\dot{\gamma}} z = 0$ - equivalent definition.

$$z|_{\gamma(t_2)} = U(t_2, t_1)\, z|_{\gamma(t_1)}$$

$$U = P\exp \int_{t_1}^{t_2} \omega^{\alpha}_{i\beta} \frac{dx^i}{dt}\, dt$$

$d_A^2 \overset{?}{=} 0 \qquad d_A \circ d_A z = d_A(dz + \omega z)$

$= d(dz) + d\omega z - \omega \wedge dz +$

$+ \omega \wedge dz + \omega \wedge \omega z = \Omega \wedge z$

Ω - curvature.