Legendre transform

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = \sum_{j=1}^{n} \frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\ddot q^j + \frac{\partial^2 L}{\partial \dot q^i \partial q^j}\dot q^j$$

Highest derivatives:

$H_L(q,\dot q)$ = Hessian for $\dot q$ has to be nondegenerate

Nondegenerate Lagrangian system.

Def $\theta_L = \sum \frac{\partial L}{\partial \dot q^i} dq^i$ - 1-form

Prop $d\theta_L$ is nondeg $\Leftrightarrow$ L nondeg. Proof: $d\theta_L = \dots$ Take $d\theta_L^n$

Def T^*M and standard coord

$(\vec p, \vec q) = (p_1, \dots, p_n, q^1, \dots, q^n)$

$p_i(df) = \frac{\partial f}{\partial q^i}$

Coord. corr. to the basis $dq^1 \dots dq^n$

Def 1-form θ on T^*M:

$\theta = \vec p \cdot d\vec q$ - Liouville canonical 1-form

$\theta(u) = p(\pi_* u) \quad u \in T_{(p,q)} T^*M$

$\pi : T^*M \to M$

Def $\tau_L : TM \to T^*M$ is call Legendre transf. for L if

$$\theta_L = \tau_L^*(\theta)$$

$\tau_L(\vec q, \dot{\vec q}) = (\vec p, \vec q) \qquad \vec p = \frac{\partial L}{\partial \dot{\vec q}}(\vec q, \dot{\vec q})$

Notice. It is diffeom. iff. L-nondeg.

Def. Take $\tau_L : TM \to T^*M$ - diffeom.

$H : T^*M \to \mathbb{R}$ - Hamiltonian

$H \circ \tau_L = E_L = \dot{\vec q}\frac{\partial L}{\partial \dot{\vec q}} - L$

$H(p,q) = p\dot q - L(q,\dot q)\big|_{p = \frac{\partial L}{\partial \dot q}}$

Thm $\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = 0 \Leftrightarrow$

$$\dot p_i = -\frac{\partial H}{\partial q^i}, \quad \dot q^i = \frac{\partial H}{\partial p_i}$$

Proof $dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq =$

$= \cancel{p\, d\dot q} + \dot q\, dp - \frac{\partial L}{\partial q}dq - \cancel{\frac{\partial L}{\partial \dot q}d\dot q}\Big|_{p = \frac{\partial L}{\partial \dot q}} =$

$= \dot q\, dp - \frac{\partial L}{\partial q}dq\Big|_{p = \frac{\partial L}{\partial \dot q}}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \dot p$

Ex. $\mathcal{L} = \frac{m\dot{\vec r}^2}{2} - V(\vec r) = T - V$

$$\vec p = \frac{\partial \mathcal{L}}{\partial \dot{\vec r}} = m\dot{\vec r}$$

$$H(\vec p, \vec r) = (\vec p \cdot \dot{\vec r} - \mathcal{L}) = \frac{\vec p^2}{2m} + V(\vec r) = T + V.$$

Hamiltonian mechanics

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i = dq^i \wedge dp_i \qquad g_t^H \text{ - Hamilt. phase flow.}$$

$$\omega(X_H) = dH$$

Thm. $g_t^*\omega = \omega$

Proof: $\frac{d}{dt} g_t^*\omega = \mathcal{L}_{X_H}\omega =$

$$= d\, i_{X_H} + i_{X_H} d\,\omega = 0$$

Def. $\mathcal{L} \subset T^*M$ is called Lagrangian if $\dim \mathcal{L} = \dim M$, $\omega|_{\mathcal{L}} = 0$

Prop. g_t^H preserves $\mathcal{L}$.

Prop. ω^n - volume form on T^*M.

Action functional. ← Poincare-Cartan form

1-form $\theta - H\,dt = p\,dq - H\,dt$ on $T^*M \times \mathbb{R}$

Thm Admissible path in $T^*M \times \mathbb{R}$ is extremal for

$$S(\gamma) = \int_\gamma (p\,dq - H\,dt) = \int_{t_0}^{t_1} (p\dot q - H)\,dt$$

iff it is a lift of a path $\gamma(t) = (\vec p(t), \vec q(t))$

$$\dot{\vec p} = -\frac{\partial H}{\partial \vec q}, \quad \dot{\vec q} = \frac{\partial H}{\partial \vec p}$$

Poisson bracket and classical observables $f \in C^\infty(T^*M)$

$$\frac{df_t}{dt} = X_H(f_t) = \frac{\partial H}{\partial p}\frac{\partial f_t}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f_t}{\partial p} = \{f_t, H\}$$

$f_t = f(g_t(p,q))$

$$\{f, g\} = X_f(g) = dg\, X_f = -\omega(X_f, X_g)$$

Properties: $\{f,g\} = -\{g,f\}$

$$\{fg, h\} = f\{g,h\} + g\{f,h\}$$

Jacobi

Integral of motion: $\{H, I\} = 0$

$$\Leftrightarrow \left.\frac{dI}{dt}\right|_{\text{eq. of motion}} = 0$$