Special relativity: brief overview

Physical axioms:

1) The laws of physics are independent of all inertial frames of reference
2) Speed of light has the same value in all inertial frames

Mathematical reformulation

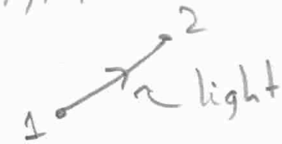
2) Spacetime is a pseudoriemannian manifold (M, g) of signature $(1,3)$. Metric g is given by Minkowski metric in the chart corresponding to inertial frame
1) Laws of physics are invariant when represented in any chart for which g is in Minkowski form.

Minkowski form: $\quad g = \{\eta_{\mu\nu}\} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Interval, Lorentz group, proper time:

$K \leftarrow$ inertial frames $\rightarrow K'$

$(x, y, z, t) \qquad (x', y', z', t')$



$|\vec{x}_2 - \vec{x}_1| = c(t_2 - t_1)$

$|\vec{x}'_2 - \vec{x}'_1| = c(t'_2 - t'_1)$

$$0 = s'_{12} = s_{12} = \left(c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1)^2\right)$$

new interval

$$ds = c^2dt^2 - dx^2 - dy^2 - dz^2 = -\eta_{\mu\nu}\, dx^\mu dx^\nu$$

where $x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$

$ds = ds'$ – invariance under linear transformations

$$\Lambda^T \eta \Lambda = \eta \qquad O(1,3) \text{ – Lorentz group}$$

components: $\Lambda^\mu{}_\nu\, \eta_{\mu\rho}\, \Lambda^\rho{}_\upsilon = \eta_{\nu\rho}$

$O(1,3)$ has 4 connected components:

$SO^\uparrow(1,3)$ – proper Lorentz group (preserves time direction)

$$O(1,3) = SO^\uparrow(1,3) \cup SO^\uparrow(1,3)P \cup SO^\uparrow(1,3)PT \cup SO^\uparrow(1,3)PT$$

$P = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \quad T = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

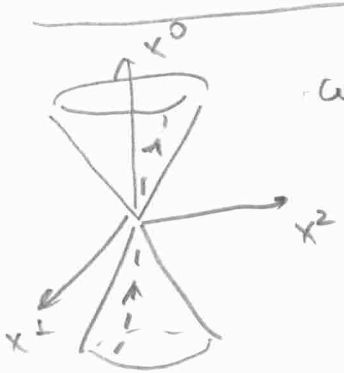
$SO^{\uparrow}(1,3)$ generated by rotations + boosts

Boosts: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\varphi & \mathrm{sh}\varphi \\ \mathrm{sh}\varphi & \mathrm{ch}\varphi \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$

Take $x' = 0 \Rightarrow \frac{x}{ct} = \mathrm{th}\varphi = \frac{v}{c}$

solving for φ:

$$t' = \frac{t + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c}}} \qquad x' = \frac{x + vt}{\sqrt{1 - v^2/c^2}}$$



worldline of a particle belongs to the inside of the lightcone:

$$x^{0\,2} = (x^1)^2 + (x^2)^2 + (x^3)^2$$

light rays travel on the boundary

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 d\tau^2$$

proper time

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}} \Rightarrow$$

(in chart, corresponding to the particle)

$$\Rightarrow \frac{dt}{\sqrt{1 - \frac{v^2}{\ldots}}} = d\tau$$

twin paradox

Relativistic particle

4-velocity: (unit tangent vector to the worldline)

$$u^\mu = \frac{dx^\mu}{ds} \qquad ds = c\,dt\sqrt{1 - \frac{v^2}{c^2}}$$

$$u = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\vec{v}}{c\sqrt{1 - \frac{v^2}{c^2}}} \right)$$

$$\langle u, u \rangle_\eta = -1$$

Action: $S = \int_{t_1}^{t_2} L\, dt = -\alpha \int_{t_1}^{t_2} ds$

$L = -mc$ why?

$$L = -\alpha c \sqrt{1 - \frac{\vec{v}^2}{c^2}} \underset{c\to\infty}{\approx} -\alpha c + \frac{\alpha \vec{v}^2}{2c} \Rightarrow \alpha = mc$$

$$S = -mc \int_{t_1}^{t_2} \sqrt{1 - \frac{\vec{v}^2}{c^2}}\, dt$$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{|\vec{v}|^2}{c^2}}}$$ - momenta

$$\mathcal{E} = \vec{p}\cdot\vec{v} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$v \to 0$ $\quad \boxed{E = mc^2}$

$v \lll c$ $\quad E = mc^2 + \frac{mv^2}{2}$

$$\mathcal{E}^2 = c^2|\vec{p}|^2 + m^2c^4$$

$$\mathcal{H} = c\sqrt{|\vec{p}|^2 + m^2c^2}$$

$p \ll mc$ $\quad \mathcal{H} \approx mc^2 + \frac{|\vec{p}|^2}{2m}$

Also, $\vec{p} = \frac{\mathcal{E}\vec{v}}{c^2}$ $\quad |\vec{v}| = c \Rightarrow E, p \to \infty$

$m \neq 0$ - no speed of light for you!

$$m = 0 \Rightarrow |\vec{p}| = \mathcal{E}/c$$

Notice: $\hat{p}^\mu = mcu^\mu = (\mathcal{E}/c, \vec{p})$

4-vector - relativistic momentum

$$\langle \hat{p}, \hat{p} \rangle_\eta = -m^2c^2$$

Relativistic particle in EM field:

$$S(\gamma) = \int_{t_0}^{t_1} \left(-mc\,ds - \frac{e}{c}\,\gamma^* A\right)$$

charge

Here $A = i\omega$, where ω - $U(1)$ connection

upper index $\vec{A}^\mu = (\varphi, \vec{A})$

$$\mathcal{L} = \frac{mv^2}{2} + \frac{e}{c}\vec{A}\cdot\vec{v} - e\varphi$$

in $c \to \infty$ limit

non-relativistic particle

Exercise:

1) $\mathcal{H} = \sqrt{m^2c^4 + c^2\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi$

Hamiltonian

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} + \frac{e}{c}\vec{A} = \vec{p} + \frac{e}{c}\vec{A}$$

non-relativistic limit:

$$\mathcal{H} = \frac{1}{2m}\left(\vec{P} - \frac{e}{c}\vec{A}\right)^2 + e\varphi$$

2) Euler-Lagrange equations:

$$m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt} = -\frac{e}{c}\frac{\partial\vec{A}}{\partial t} - e\vec{\nabla}\varphi + \frac{e}{c}[\vec{v} \times \mathrm{curl}\,\vec{A}]$$

$$\Updownarrow$$

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}[\vec{v} \times \vec{B}]$$

Gauge invariance

$A \to A + d\Lambda$ naturally gauge-invariant

Classical Field Theory.

Maxwell equations: (c=1 from now on)

U(1) connection ω: $A = i\omega$ on $\mathbb{R}^{1,3}$

$F = dA$

$$\begin{cases} dF = 0 & \leftarrow \text{Bianchi identity} \\ d*F = 0 \end{cases}$$

Maxwell equations in vacuum

$A^\mu = (\varphi, \vec{A})$

$$\begin{aligned}
&\operatorname{div} \vec{E} = 0 \\
&\operatorname{div} \vec{B} = 0 \\
&\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
&\operatorname{curl} \vec{B} = \frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

manifestly Lorentz-inv. form

Action:

$$S = -\frac{1}{4}\int F \wedge * F =$$

$$= -\frac{1}{4}\int \underbrace{F_{\mu\nu} F^{\mu\nu}}_{\text{Lagrangian density}} \mathrm{vol}_M$$

$$= \int \mathcal{L}(A_\mu, \partial_\nu A_\mu)\, \mathrm{vol}_M$$

locally diff. polynomial in A

trip from classical mechanics to field theory:

Paths: $[t_0, t_1] \xrightarrow{\gamma} M$ $\quad \{q^i(t)\}$

Fields: 1) sections of vector bundles (matter fields)

2) connections on vector bundles (interaction mediators)

More elementary example:

$$(\Delta - m^2)\,\phi(x) = 0 \qquad \text{Klein-Gordon equation}$$

$$\Delta = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \vec{x}^2}\right) = \eta^{\mu\nu}\partial_\mu\partial_\nu$$

$$S = \frac{1}{2}\int_M \phi\Delta\phi - m^2\phi^2\, \mathrm{vol}_M =$$

$$= -\frac{1}{2}\int (d\phi \wedge * d\phi + m^2\phi^2\, \mathrm{vol}_M) =$$

$$= -\frac{1}{2}\int \mathrm{vol}_M (\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2)$$

E-L equations: (for simplicity in $\mathbb{R}^{1,3}$)

$$S = \int_{\mathbb{R}^{1,3}} d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \qquad \int \delta\phi \frac{\delta \mathcal{L}}{\delta \phi} \quad \shortparallel$$

Frechet derivative: $F[\phi + \delta\phi] = F[\phi] + F_\phi[\delta\phi] + \ldots$