Relative homotopy groups (Lecture VII)

(D^i, S^{i-1}, y_0), (X, A, x_0),

where $y_0 \in S^{i-1} = \partial D^i \subset D^i$ and $x_0 \in A \subset X$

$\boxed{i \geq 2}$

The <u>map of triples of spaces</u>

$f : (D^i, S^{i-1}, y_0) \to (X, A, x_0)$

such that $f : D^i \to X$ and $f(S^{i-1}) \subset A$, $f(y_0) = x_0$

f_t, $t \in [0,1]$, connecting f_0, f_1 is called homotopy of triples if for any t it is a map of triples.

$\pi_i(X, A, x_0)$ is a group.

the set of homotopy classes of maps above

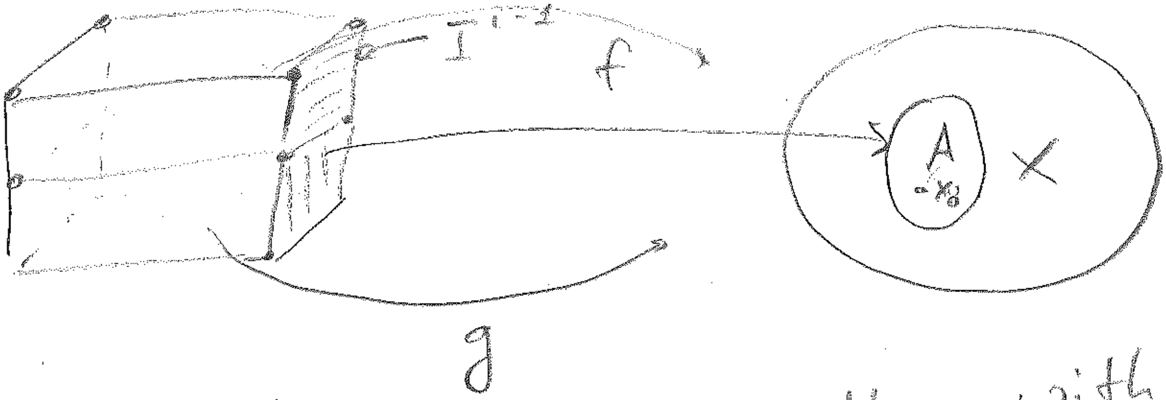
<u>Group structure:</u> Consider $(I^i, \partial I^i, J_i)$

$I = [0,1]$ $J_i = \partial I^i - \overset{\circ}{I}{}^{i-1}$ one face

Consider $\{(I^i, \partial I^i, J_i) \to (X, A, x_0)\}$ $(*)$

(D^i, S^{i-1}, y_0) is homotopically eq. to $(I^i, \partial I^i, J_i)$

and homotopy classes of $(*)$ are in 1-1 corr. with $\pi_i(X, A, x_0)$



Exercise In $i=1$ case - problems with group structure Explain why.

Exercise Show that if $i \geq 3$ $\pi_i(X, A, x_0)$ is abelian

Unit element: class of $f : D^i \to x_0$

It is enough to show that $f(D^i) \subset A$

Exact homotopy sequence of a pair

(X, A) - pair $\pi_i(A), \pi_i(X), \pi_i(X, A)$

Let's look at the maps between those

$\pi_i(A) \to \pi_i(X)$ (obviously)

$(S^i, y_0) \to (X, x_0) \longmapsto (D^i, S^{i-1}, y_0) \to (X, x_0, x_0)$

(we represent S^i as a factorspace D^i/S^{i-1})

As a result: $\pi_i(X) \to \pi_i(X, A)$

Finally, for the map $(D^i, S^{i-1}, y_0) \to (X, A, x_0)$

we can consider the map $(S^{i-1}, y_0) \to (A, x_0)$

As a result we get $\pi_i(X, A) \to \pi_{i-1}(A)$

$$\to \pi_i(X,A) \to \pi_{i-1}(A) \to \pi_{i-1}(X) \to \pi_{i-1}(X,A) \to \pi_{i-2}(A)$$

It ends up with maps of sets:

$$\pi_1(X) \to \pi_1(X,A) \to \pi_0(A) \to \pi_0(X)$$

group

$S^0 \to A$ connected

$S^0 \to X$ components

Def. A sequence of groups and homomorphisms is exact in G_i

$$G_1 \to G_2 \to \dots \to G_i \to \dots$$

if the image of $G_{i-1} \to G_i$ coincides with the kernel of $G_i \to G_{i+1}$.

If the whole sequence is exact, then one can deduce something about G_i's

$0 \to G_{i+1} \to G_{i+2} \to 0$ – isomorphism

$0 \to G_{i+1} \to G_{i+2} \to G_{i+3} \to 0$ $\quad G_{i+3} = G_{i+2}/G_{i+1}$

Proposition 5-lemma

exact

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "\varphi_1"] & G_2 \arrow[r] \arrow[d, "\varphi_2"] & G_3 \arrow[r] \arrow[d, "\varphi_3"] & G_4 \arrow[r] \arrow[d, "\varphi_4"] & G_5 \arrow[d, "\varphi_5"] \\
H_1 \arrow[r] & H_2 \arrow[r] & H_3 \arrow[r] & H_4 \arrow[r] & H_5
\end{tikzcd}$$

exact

φ_1 – epimorphism, φ_2, φ_4 – isom, φ_5 – monomorph.

Then φ_3 - isomorphism

Theorem Homotopical sequence of (X, A) is exact

Proof Check all six homomorphism combinations.
Check 3, all others are the exercise

Example Let $(X, A) \to (Y, B)$ induce isomorphism of $\pi_i(X) \to \pi_i(Y)$ and $\pi_i(A) \to \pi_i(B)$
Show that this map induces isomorphism on all relative homotopy groups

Example Concider the pair $\underset{\text{cone}}{CX}, X$