## Spinor fields

$SO^{\uparrow}(1,3)$ (det = 1, preserves dir. of time)

Full Lorentz group: $SO^{\uparrow}(1,3) \rtimes \{1, P, T, PT\}$ (P: parity, T: time reflection)

Notation:

$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \Lambda^{\mu}{}_{\alpha} \eta_{\mu\nu} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

$(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0 \geq 1$

that's why two comp. $\uparrow \downarrow$

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$SL(2,\mathbb{C}) \cong \mathrm{spin}(3,1) = \widetilde{SO}(3,1)$

$$\hat{x} = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} = x^{\mu} \sigma_{\mu}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prop. 1) $\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k$, $\mathrm{Tr}(\sigma_{\mu} \sigma_{\nu}) = 2\delta_{\mu\nu}$

2) $x^{\mu} = \frac{1}{2} \mathrm{Tr}(\hat{x} \sigma_{\mu})$ 3) $x^{\mu} x_{\mu} = -\det \hat{x}$

$SL(2,\mathbb{C})$ transformation:

$$\hat{x}' = A \hat{x} A^{\dagger} \qquad \det(\hat{x}') = \det(x)$$

$$\Lambda^{\mu}{}_{\nu} \sigma_{\nu} = A \sigma_{\mu} A^{\dagger}$$

Lorentz transformation

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$S_+$, $S_-$ – Spinor repres. of $\mathrm{spin}(3,1)$

$S_+ : A \longrightarrow A$, $S_- : A \longrightarrow (A^{\dagger})^{-1}$

Van der Waerden notation:

$\psi_{\alpha} \to A_{\alpha}^{\beta} \psi_{\beta}$ (2-component vector in $S_+$) $\qquad \bar{\psi}^{\dot{\beta}} \to (A^{\dagger -1})^{\dot{\alpha}}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}$ (2-component vector in $S_-$)

Pairing: $\epsilon^{\alpha\beta} \psi_{\alpha} \chi_{\beta} = \psi \wedge \chi$

$\epsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} = \bar{\psi} \wedge \bar{\chi}$

Dirac spinors: $S_+ \oplus S_-$

$$A \to \begin{pmatrix} A & 0 \\ 0 & (A^{\dagger})^{-1} \end{pmatrix}$$

4-dim vector: $(\{\psi_{\alpha}\}, \{\bar{\chi}^{\dot{\alpha}}\})$

$SL(2,\mathbb{C}) \cong \widetilde{SO}(3,1) = \mathrm{spin}(3,1)$

Spinor representations:

Left: $S_+$ Right: $S_-$

$S_+ : A(\Lambda) \to A(\Lambda)$ $S_- : A(\Lambda) \to A^{\dagger -1}(\Lambda)$

↑ lift

These are known as Weyl spinors.

$S_+ \oplus S_-$ $S(\Lambda) = \begin{pmatrix} A(\Lambda) & 0 \\ 0 & A^{\dagger -1}(\Lambda) \end{pmatrix}$

↑ Dirac spinor

Effectively, this repres. can be constructed as follows:

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$ – Clifford algebra

$[\gamma^\mu, \gamma^\nu]$ – satisfy comm. relations of Lie algebra $so(3,1)$

↑ commutator

$\gamma^0 = \begin{pmatrix} 0 & -i\sigma_0 \\ -i\sigma_0 & 0 \end{pmatrix}$ $\gamma^i_{i=1,2,3} = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ← distinguishes L, R spinors

irreducible representation of Clifford algebra

In general: $0 \to \mathbb{Z}_2 \to \mathrm{Spin}(p,q) \to SO(p,q) \to 0$



$S(\Lambda)^{-1} = -\gamma^0 S^\dagger(\Lambda)\gamma^0$

$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu \Rightarrow \gamma^\mu$ – behaves like vector

Dirac conjugate $\overline{\Psi} = i\overline{\Psi}\gamma^0 \Rightarrow$

$\Rightarrow \overline{\chi}\psi$ – invariant

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Dirac operator on $\mathbb{R}^{1,3}$:

$\gamma^\mu \partial_\mu + m$ – Lorentz invariant

Consider a spin(1,3) (Dirac) spinor bundle over $\mathbb{R}^{1,3}$ $\psi(x)$

$S = \int d^4x\, \overline{\Psi}(x)(\gamma^\mu\partial_\mu + m)\Psi(x)$

Lorentz invariant

$P = i\gamma^0$ (since $\gamma^0\gamma^i(\gamma^0)^{-1} = -\gamma^i$)

$T = (CP)^{-1}\gamma_5\gamma_0$

$C = \gamma_2\gamma_0$ : $\gamma_\mu^T = -C\gamma_\mu C^{-1}$

$S = \int d^4x\, \overline{\Psi}(x)(\gamma^\mu(\partial_\mu + eA_\mu) + m)\Psi(x)$

$\psi^c = C\overline{\Psi}^T$

$= \int d^4x\, \overline{\Psi}^c(x)(\gamma^\mu(\partial_\mu - eA_\mu) + m)\Psi^c(x)$

$$\mathrm{Spin}^c(3,1) = \mathrm{Spin}(3,1)\times_{\mathbb{Z}_2} S^1$$
$$(a,u) \quad (-a,-u)$$

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Dirac equation on a manifold.

<u>Spin connection:</u> $g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}$

$\Updownarrow$

Orthonormal frame: $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$

$$\omega_\mu{}^a{}_b = e^a{}_\nu \Gamma^\nu_{\lambda\mu} e^\lambda_b + e^a{}_\nu \partial_\mu e^\nu{}_b$$

Christoffel symbol

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} = 0$$

$\omega_{\mu(ac)}$ antisymm.

$$\omega_\mu \to \omega_\mu{}_{ab}[\gamma_a,\gamma_b]$$

connection on spinor bundle

Dirac operator $\not{D} = \gamma^\mu \partial_\mu \to \gamma^a e^\mu_a \nabla_\mu$

$$S = \int e\, \bar{\Psi}(\not{D} + m)\Psi$$

$$\sqrt{-g} = \det e^a_\mu$$

<u>Extra:</u>

1) General Dirac operator:

<u>Generalized clifford module:</u>

$$c : \Gamma(T^*M \otimes E) \to \Gamma(E)$$
$$v^* \otimes s \to c(v^*)s$$
$$c(v^*)c(w^*) + c(w^*)c(v^*) = 2g(v^*, w^*)$$

<u>Clifford connection:</u>

$$\nabla^E_X(c(v^*)s) = c(\nabla_X v^*)s + c(v^*)\nabla^E_X s$$

$$\not{D} : \Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M\otimes E) \xrightarrow{c} \Gamma(E)$$

$$\not{D} = c(e_i)\nabla^E_{e_i} \quad \text{- locally}$$

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<u>Ex.</u> $\quad E = \Omega^\cdot(M)$

$$c(v^*)\alpha = v^* \wedge \alpha - i_v \alpha, \quad v = g(v^*, \cdot)$$

$$\underline{\not{D} = d + \delta}$$