Quantum Field Theory

$$S = -\frac{1}{2}\int d^4x \,\underbrace{(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2)}_{=\mathcal{L}(x)}$$

$$(-\partial_\mu\partial^\mu + m^2)\phi = 0$$

$$\mathcal{H}(x) = \frac{1}{2}\pi\dot\phi - \mathcal{L} =$$
$$= \frac{1}{2}\pi^2 + (\vec\nabla\phi)^2 + m^2\phi^2$$

$$[\phi(\vec x,t),\pi(\vec y,t)] = i\delta^{(3)}(\vec x - \vec y)$$

$$[\phi(\vec x,t),\phi(\vec y,t)] = [\pi(\vec x,t),\pi(\vec y,t)] = 0$$

Heisenberg picture

operator-valued function on $\mathbb{R}^4$

$$\phi(f,t) = \int d^3x\, f(\vec x)\phi(\vec x,t)$$
$$\pi(f,t) = \int d^3x\, g(\vec x)\pi(\vec x,t)$$

$$[\phi(f,t),\pi(g,t)] = i\int d^3x\, f(\vec x) g(\vec x)$$

$$\phi(x) = \int d(\vec k)\left[a(\vec k)e^{i(\vec k\cdot\vec x - E_k t)} + a^\dagger(\vec k)e^{-i(\vec k\cdot\vec x - E_{\vec k}t)}\right]$$

complex conjugated

$$E_{\vec k} = \sqrt{|\vec k|^2 + m^2}$$

$$k_\mu k^\mu + m^2 = 0$$

$$\int (dk) = \int \frac{d^3\vec k}{(2\pi)^3 2E_k} \Leftrightarrow \int\frac{d^4k}{(2\pi)^4}2\pi\delta(k^2+m^2)\theta(k^0)$$

$$[a(\vec k), a^\dagger(\vec p)] = (2\pi)^3 2E_{\vec k}\delta^3(\vec k - \vec p)$$

$$[a(\vec k), a(\vec p)] = [a^\dagger(\vec k), a^\dagger(\vec p)] = 0$$

$$H = \frac{1}{2}\int (dk) E_{\vec k}[a^\dagger(\vec k)a(\vec k) + a(\vec k)a^\dagger(\vec k)]$$

subtract infinite term

$$H = \int (dk) E_{\vec k}\, a^\dagger(\vec k)a(\vec k)$$

$$P^\mu = (H,\vec P) = \int (dk) k^\mu a^\dagger(\vec k)a(\vec k)$$

$$[\phi(x), P^\mu] = i\partial^\mu\phi(x)$$

$$N = \int (dk)\, a^\dagger(\vec k)a(\vec k)$$

$$\frac{1}{n!}\int (dk_1)\dots(dk_n) F_n(\vec k_1,\dots \vec k_n) a^\dagger(\vec k_1)\dots a^\dagger(\vec k_n)|0\rangle$$

$$\langle F, G\rangle = F_0^* G_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\int (dk_1)\dots(dk_n) F_n^*(\vec k_1,\dots,\vec k_n) G_n(\vec k_1,\dots,\vec k_n)$$

Ω – sequences with finitely many zero entries, F decreasing fast

Lemma

$$\int d^4p\, \delta(p^2+M^2)\,\Theta(p^0)\, f(p) =$$

$$= \int d^3\vec{p}\, dp^0\, \delta((p^0)^2 - \vec{p}^2 - M^2)\,\theta(p^0)\, f(p)$$

$$= \int d^3\vec{p}\, \frac{f(\vec{p}, \sqrt{\vec{p}+M^2})}{2\sqrt{\vec{p}+M^2}}$$

Use property: $\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{\left|\frac{df}{dx}(a_i)\right|}$

a_i - roots.

Operators, ϕ, a, a^+ : $\Omega \to \Omega^*$

$$Z(J) = N \int [D\phi]\, e^{iS} =$$

$$S = \int d^4x \left[\frac{1}{2}\phi(\Delta - m^2)\phi + J(x)\phi(x) \right]$$

$$(\Delta + m^2)\, G(x-y) = \delta^4(x-y)$$

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-1}{k^2 - m^2 + i\epsilon} e^{ik(x-y)}$$

$$Z(J) = \exp \frac{i}{2} \int d^4x\, d^4y\, J(x)\, G(x-y)\, J(y)$$

$$\langle 0 | T(\phi(x_1) \ldots \phi(x_n)) | 0 \rangle = (-i)^n \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_n)} Z(J) \Big|_{J=0}$$

Back to origins.

Elementary particles

Elementary particles

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$$\Lambda^\mu{}_\nu \qquad (\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho = \eta_{\nu\mu}\eta^{\rho\sigma}\Lambda^\mu{}_\sigma$$

Poincare group

$$U(\bar\Lambda, \bar a)\, U(\Lambda, a) = U(\bar\Lambda\Lambda, \bar\Lambda a + \bar a)$$

Quantum states $\Psi_{p,\sigma}$:

$$P^\mu \Psi_{p,\sigma} = p^\mu \Psi_{p,\sigma}$$

$$U(1,a) = e^{-ia_\mu P^\mu} \Rightarrow U(1,a)\Psi_{p,\sigma} = e^{-ipa}\Psi_{p,\sigma}$$

$$P^\mu U(\Lambda)\Psi_{p,\sigma} = U(\Lambda)\left[U^{-1}(\Lambda) P^\mu U(\Lambda)\right]\Psi_{p,\sigma} =$$

$$= U(\Lambda)(\Lambda^{-1})_\rho{}^\mu P^\rho \Psi_{p,\sigma} =$$

$$= \Lambda^\mu{}_\rho\, p^\rho U(\Lambda)\Psi_{p,\sigma}$$

Therefore: $U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\Psi_{\Lambda p,\sigma'}$

$$p^2 = \eta_{\mu\nu}p^\mu p^\nu \qquad \underbrace{p^2 \le 0 \qquad \mathrm{sgn}(p^0)}_{\text{inv.}}$$

$$p^\mu = L^\mu{}_\nu(p)\, k^\nu$$

L - standard Lorentz transformation:

$$\Psi_{\vec p,\sigma} = N(p)\, U(L(p))\, \Psi_{k,\sigma}$$

$N(p)$ normalization factor

$$U(\Lambda)\Psi_{p,\sigma} = N(p)\, U(\Lambda L(p))\Psi_{k,\sigma} =$$

$$= N(p)\, U(L(\Lambda p))\, \underbrace{U(L^{-1}(\Lambda p)\Lambda L(p))}_{\text{leaves } k \text{ invariant}}\Psi_{k,\sigma}$$

$$W^\mu{}_\nu k^\nu = k^\mu \quad \text{Little group}$$

$$U(W)\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma'\sigma}(W)\Psi_{k,\sigma'}$$

Representation of little group

$$U(\Lambda)\Psi_{p,\sigma} = N(p)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\, U(L(\Lambda p))\Psi_{k,\sigma'}$$

$$U(\Lambda)\Psi_{p,\sigma} = \left(\frac{N(p)}{N(\Lambda p)}\right)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma'}$$

$$k = (0,0,0,\underbrace{M}_{k^0}) \qquad (0,0,\varkappa,\varkappa)$$

$$SO(3) \qquad ISO(2)$$

$$(\Psi_{p',\sigma'}, \Psi_{p,\sigma}) = |N(p)|^2 \delta_{\sigma'\sigma}\delta^3(\vec k' - \vec k)$$

$$(\Psi_{k',\sigma'}, \Psi_{k,\sigma}) = \delta^3(\vec k' - \vec k)\delta_{\sigma'\sigma}$$

$$\int d^4p\, \delta(p^2+M^2)\, \theta(p^0)\, f(p) =$$

$$= \int d^3\vec{p}\; \frac{f(\vec{p}, \sqrt{\vec{p}^2+M^2})}{2\sqrt{\vec{p}^2+M^2}}$$

Invariant volume element: $\dfrac{d^3\vec{p}}{\sqrt{\vec{p}^2+M^2}}$

$$F(\vec{p}) = \int F(\vec{p}')\, \delta^3(\vec{p}-\vec{p}')\, d^3\vec{p}' =$$

$$= \int F(\vec{p}')\left(\sqrt{|\vec{p}'|^2+M^2}\, \delta^3(\vec{p}'-\vec{p})\right) \frac{d^3\vec{p}'}{\sqrt{\vec{p}'^2+M^2}}$$

$$\sqrt{|\vec{p}'|^2+M^2}\, \delta^3(\vec{p}'-\vec{p}) = p^0 \delta^3(\vec{p}'-\vec{p})$$

$$p^0 \delta^3(\vec{p}'-\vec{p}) = k^0 \delta^3(\vec{k}-\vec{k})$$

Lorentz invariance

$$(\Psi_{p',\sigma'}, \Psi_{p,\sigma}) = |N(p)|^2\, \delta_{\sigma'\sigma} \frac{p^0}{k^0}\, \delta^3(\vec{p}'-\vec{p})$$

$$N(p) = \sqrt{\frac{k^0}{p^0}} \quad \text{- normalization}$$

(can be $N(p) = \sqrt{\frac{k^0}{p^0}}$)

$$(\Psi_{p',\sigma'}, \Psi_{p,\sigma}) = \delta_{\sigma',\sigma}\, \delta^3(\vec{p}'-\vec{p})$$

Cases: 1) $M>0$

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))\, \Psi_{\Lambda p,\sigma'}$$

Wigner rotation

2) $M=0$

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}\, e^{i\sigma\theta(\Lambda,p)}\, \Psi_{\Lambda p,\sigma}$$

$\dfrac{\vec{J}\cdot\vec{p}}{|\vec{J}\cdot\vec{p}|}$ - helicity $\quad J_3$

$$\Psi_{\vec{p},\sigma} = a(\vec{p},\sigma)|0\rangle$$

P, T: $\quad P\, iH P^{-1} = iH$

$$T\, iP^\rho T^{-1} = i\, \mathcal{T}_\mu{}^\rho P^\mu$$

$$T\, iH T^{-1} = -iH$$

If T - unitary $\quad THT^{-1} = -H$

$\Rightarrow |E\rangle,\ T|E\rangle$

$\qquad\qquad E \quad -E$

$\underline{U}$

T - antiunitary $\Rightarrow (Tx, Ty) = \overline{(x,y)}$

$$|\ldots p, \sigma, n, \ldots, p', \sigma', n' \rangle =$$

$$= \alpha_n | \ldots q', \sigma', n', \ldots p, \sigma, n \ldots \rangle$$

$\alpha_n^2 = 1$ bosons and fermions

$$\langle \Phi_{q'} | \Phi_q \rangle = \delta(q'-q) = \delta^{(3)}(\vec{p}'-\vec{p})\delta_{\sigma'\sigma}\delta_{n'n}$$

$$\langle \Phi_{q_1', q_2'} | \Phi_{q_1, q_2} \rangle = \delta(q_1'-q_1)\delta(q_2'-q_2) \pm \delta(q_2'-q_1)\delta(q_1'-q_2)$$

$$a(q')a^\dagger(q) \mp a^\dagger(q)a(q') = \delta(q-q')$$

$$\mathcal{O} = \sum_{N=0}^{\infty}\sum_{M=0}^{\infty} \int dq_1' \ldots dq_N' \, dq_1 \ldots dq_M \; a^\dagger(q_1') \ldots a^\dagger(q_N') \, a(q_1) \ldots a(q_M) \; C_{NM}(q_1' \ldots q_N'; q_1, \ldots, q_M)$$

Cluster decomposition principle

$$S^c_{q',q} = S_{q',q} = \delta(q'-q)$$

$$S_{q_1' q_2', q_1 q_2} = \underbrace{S^c_{q_1' q_2', q_1, q_2}}_{\text{connected part}} + \delta(\;)\text{-functions}$$

$$S_{\beta\alpha} = \sum_{\text{part}} (\pm) \, S^c_{\beta_1\alpha_1} S^c_{\beta_2\alpha_2} \ldots$$

$$\tilde{S}^c_{x_1' x_2' \ldots x_1, x_2 \ldots} =$$

$$\boxed{\text{RL theorem: } \hat{f}(\lambda) = \int f(x) e^{-i\lambda x} dx \to 0 \quad |\lambda| \to \infty}$$

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$$= \int d^3p_1' \, d^3p_2' \ldots d^3p_1 \, d^3p_2 \ldots S^c_{\vec{p}_1' \vec{p}_2' \ldots \vec{p}_1 \vec{p}_2 \ldots} \; e^{i\vec{p}_1'\vec{x}_1'} e^{i\vec{p}_2'\vec{x}_2'} \ldots e^{-i\vec{p}_1\vec{x}_1} e^{-i\vec{p}_2\vec{x}_2}$$

If S^c_p - well behaved, Riemann-Lebesque thm implies S^c vanishes if any comb. of spat. coord. go to infinity. - too strong

$$S^c \sim \delta^{(3)}(\vec{p}_1' + \vec{p}_2' + \ldots - \vec{p}_1 - \vec{p}_2 - \ldots) \, \delta(E_1' + E_2' + \ldots - E_1 - E_2 - \ldots) \, C_{\vec{p}_1' \vec{p}_2' \ldots \vec{p}_1 \vec{p}_2}$$

Translation invariance

single!

by pert. theory

smooth outside of certain poles and branch cuts

$$H = \sum_{N=0}^{\infty}\sum_{M=0}^{\infty} \int dq_1' \ldots dq_N' \, dq_1 \ldots dq_M \; a^\dagger(q_1') \ldots a^\dagger(q_N') \, a(q_M) \ldots a(q_1) \; h_{N,M}(q_1' \ldots q_N', q_1, \ldots, q_M)$$

contains $\delta^{(3)}$-function

S-matrix

$$\int_{-\infty}^{\infty} dt_1 \ldots dt_n \langle \Phi_\beta | T(V(t_1)\ldots V(t_n)) | \Phi_\alpha \rangle$$

$$= \frac{n!}{n_1! n_2! \ldots n_\nu!} \sum_{\text{Part}} (\pm) \sum_{\substack{n_1 \ldots n_\nu \\ n_1 + \ldots + n_\nu = n}} \prod_{j=1}^{\nu}$$

$$\int dt_{j_1} \ldots dt_{j n_j} (\Phi_{\beta_j}, T(V(t_{j_1}) \ldots V(t_{j n_j})) \Phi_{\alpha_j})_C$$

Since

$$(\Phi_\beta, T(V(t_1) \ldots V(t_n)), \Phi_\alpha) =$$

$$= \sum_{\text{clusterings}} (\pm) \prod_{j=1}^{\nu} (\Phi_{\beta_j}, T(V_{t_{j_1}} \ldots V_{j n_j}) \Phi_{\alpha_j})_C$$

$$V(t) = \int d^3x \; \mathcal{H}(\vec{x}, t)$$ interaction picture

free $$U_0(\Lambda, a) \mathcal{H}(x) U_0^{-1}(\Lambda, a) = \mathcal{H}(\Lambda x + a)$$

time ordering is Lorentz-invariant

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0 \quad (x - x')^2 \geq 0$$

casuality - consequence of Lorentz invariance.

Fields:

$$(2\pi)^{-3/2} e^{i p_\mu x^\mu} u_e(\vec{p}, \sigma, n)$$ coeff function

$$\psi_e^+(x) = \sum_{\sigma, n} \int d^3p \; u_e(x; \vec{p}, \sigma, n) \, a(\vec{p}, \sigma, n)$$

$$\psi_e^-(x) = \sum_{\sigma, n} \int d^3p \; v_e(x; \vec{p}, \sigma, n) \, a^\dagger(\vec{p}, \sigma, n)$$

$$U(\Lambda, a) \psi_e^\pm(x) U^{-1}(\Lambda, a) =$$

$$= \sum_{\bar{e}} D_{e\bar{e}}(\Lambda^{-1}) \psi_{\bar{e}}^\pm(\Lambda x + a)$$

invariant

$$\mathcal{H}(x) = \sum g_{e_1' \ldots e_n' e_1 \ldots e_M} \psi_{e_1'}^-(x) \ldots \psi_{e_N'}^-(x) \psi_{e_1}^+(x) \ldots \psi_{e_M}^+(x)$$

$$[\psi_e^+(x), \psi_{\bar{e}}^-(y)]_\mp = (2\pi)^{-3} \sum_{\sigma, n} \int d^3p \, u_e v_e \, e^{ip(x-y)}$$

does not vanish

$$\psi_e = \kappa_e \psi_e^+(x) + \lambda_e \psi_e^-(x)$$ $(x-y)$ space-like

$$[\psi_e(x), \psi_{e'}(y)]_\mp = [\psi_e(x), \psi_{e'}^\dagger(y)]_\mp = 0.$$

$$V = \int d^3x \; :\mathcal{F}(\psi(x), \psi^\dagger(x)):$$

Axioms

1) QM: States and observables of a q.f.t. are represented by vectors and operators in a rigged Hilbert space $\mathcal{H}$.

2) Rel. inv: A continuous unitary representation of the double cover of Poincare group act in Hilbert space ($U(\Lambda, a)$) representing $x' = \Lambda x + a$

$P^\mu : U(1, a) = e^{i P^\mu a_\mu}$

P^μ — momentum op.

Eigenvalues of $P^\mu \in \{ (k^\mu)^2 \leq 0 \}$

$\exists !$ Poincare-inv. vector $|0\rangle$.

3) Fields: f - test function on $\mathbb{R}^4$

set of operators $\phi_i[f] = \int d^4x\, f(x)\, \phi_i(x)$

All these ops and their adjoints are defined on common domain $\Omega \subset \mathcal{H}$ (dense) closed under their action and that of $U(\Lambda, a)$. It is a linear set containing $|0\rangle$.

All matrix elements are distribution in f $\langle \psi | \phi_i[f] | \psi' \rangle = \langle F^{\psi, \psi'}_{\phi_i}, f \rangle$

$$U(\Lambda, a)\, \phi_i[f(x)]\, U^{-1}(\Lambda, a) =$$
$$= \sum D_{ee'}(\Lambda^{-1})\, \phi_{e'}[f(\Lambda^{-1}(x-a))]$$

$D_{ee'}$ — representation of $SL(2, \mathbb{C})$

4) Microscopic causality:

If supp f, g are space-like $(x-y)^2 > 0$ $\forall\, x \in \mathrm{supp}(f)$, $y \in \mathrm{supp}(g)$

$$[\phi_i[f], \phi_j[g]]_\pm = \phi_i[f]\phi_j[g] \pm \phi_j[g]\phi_i[f] = 0$$

"−" sign - must for observables.

5) ((Cyclicity of vacuum)) The set of vectors obtained by applying polynomials in $\phi_i[f]$ to $|0\rangle$ is dense in Ω $\Rightarrow$ dense in $\mathcal{H}$

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Dirac field $\qquad \bar\Psi = \Psi^+ \beta$

$\beta = i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\mathcal{L} = -\bar\Psi(\gamma^\mu\partial_\mu + m)\Psi$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot\Psi} = -\bar\Psi\gamma^0$$

$$H = \int d^3x(\Pi\dot\Psi - \mathcal{L}) =$$

$$\int d^3x\,(\Pi\gamma^0[\vec\gamma\cdot\vec\nabla + m]\Psi)$$

$$\dot\Psi = \frac{\delta H}{\delta \Pi} = \gamma_0(\vec\gamma\cdot\vec\nabla + m)\Psi$$

$$\Updownarrow$$

$$(\gamma^\mu\partial_\mu + m)\Psi = 0$$

$$\Psi(x) = (2\pi)^{-3/2}\int d^3p \sum_{\sigma=\pm 1/2}\{u(\vec p,\sigma)e^{-ipx}b(\vec p,\sigma) + v(\vec p,\sigma)e^{ipx}d^+(\vec p,\sigma)\}$$

$p^0 \equiv \sqrt{\vec p^2 + m^2}$

$$(-i\gamma^\mu p_\mu + m)u(\vec p,\sigma) = 0$$

$$(i\gamma^\mu p_\mu + m)v(\vec p,\sigma) = 0$$

$\left\{\begin{array}{l} u(\vec p,\sigma) \\ v(\vec p,\sigma)\end{array}\right.$ 4 eigenvectors of $i\gamma^\mu p_\mu$ $\pm m$ since $(i\gamma^\mu p_\mu)^2 = -p^2 = m^2$

Normalization:
$\bar u(\vec p,s)u(\vec p,s') =$
$= -\bar v(\vec p,s)v(\vec p,s') = 2m\delta_{ss'}$

$$H = \sum_s \int d^3p\, p^0[b^+(\vec p,s)b(\vec p,s) - d(\vec p,s)d^+(\vec p,s)]$$

$$[b(\vec p,s), b^+(\vec p',s')]_\pm = [d(\vec p,s), d^+(\vec p',s')]_\pm = \delta_{ss'}\delta^{(3)}(\vec p - \vec p')$$

interchange d, d^+

Choice of commutators:

$b^+b - d^+d \Rightarrow$ creating more d-particles lowers energy - no lower bound

Therefore we have to choose anticommutators

$j^\mu = \bar\Psi\gamma^\mu\Psi \qquad \Psi \to e^{i\alpha}\Psi$

$$Q = e\sum_s \int d^3p\,[b^+(\vec p,s)b(\vec p,s) - d^+(\vec p,s)d(\vec p,s)]$$

Massless particle ± 1 helicity:

$$U(\Lambda)a_\mu(x)U^{-1}(\Lambda) = \Lambda_\mu^{\ \nu}a_\nu(\Lambda x) + \partial_\mu\Omega(x,\Lambda)$$