

Penner coordinates on super-Teichmüller spaces

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Outline

Introduction

Cast of characters

Coordinates on
Super-Teichmüller
space

$\mathcal{N} = 2$
Super-Teichmüller
theory

Open problems

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$\mathcal{N} = 2$ Super-Teichmüller theory

Open problems

Let $F_s^g \equiv F$ be the Riemann surface of genus g and s punctures.
We assume $s > 0$ and $2 - 2g - s < 0$.

Teichmüller space $T(F)$ has many incarnations:

- ▶ $\{\text{complex structures on } F\}/\text{isotopy}$
- ▶ $\{\text{conformal structures on } F\}/\text{isotopy}$
- ▶ $\{\text{hyperbolic structures on } F\}/\text{isotopy}$

Representation-theoretic definition:

$$T(F) = \text{Hom}'(\pi_1(F), PSL(2, \mathbb{R}))/PSL(2, \mathbb{R}),$$

where Hom' stands for Homs such that the group elements corresponding to loops around punctures are parabolic ($|\text{tr}| = 2$).

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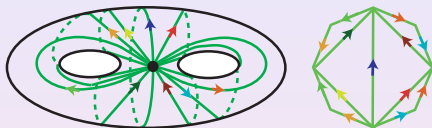
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Penner's work in the 1980s: a construction of coordinates associated to the ideal triangulation of F :



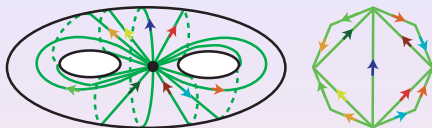
so that one assigns one positive number for every edge.

This provides coordinates for the decorated Teichmüller space:

$$\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$$

- Positive parameters correspond to the "renormalized" geodesic lengths
- \mathbb{R}_+^s -fiber provides cut-off parameter (determining the size of the horocycle) for every puncture.

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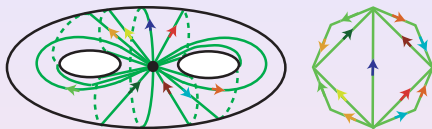
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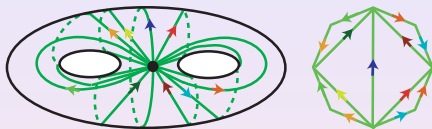
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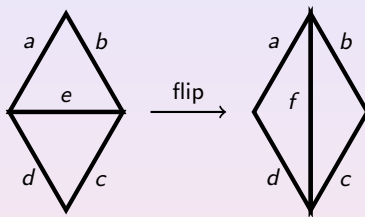
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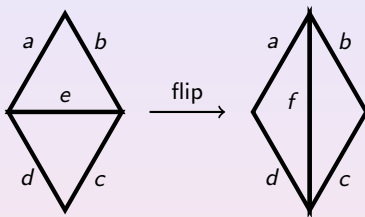
The action of $MC(F)$ can be described combinatorially using elementary transformations called flips:



Ptolemy relation : $ef = ac + bd$

In order to obtain coordinates on $T(F)$, one has to consider *shear coordinates* $z_e = \log(\frac{ac}{bd})$, which are subjects to certain linear constraints.

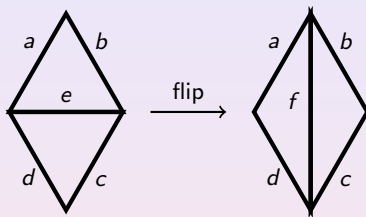
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$\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Teichmüller spaces $ST(F)$, related to supergroups $OSP(1|2)$, $OSp(2|2)$ correspondingly. In the late 80s the problem of construction of Penner's coordinates on $ST(F)$ was introduced on Yu.I. Manin's Moscow seminar.

The $\mathcal{N} = 1$ case was solved nearly 30 years later in:

R. Penner, A. Zeitlin, [arXiv:1509.06302](#).

The $\mathcal{N} = 2$ case is solved recently in:

I. Ip, R. Penner, A. Zeitlin, [arXiv:1605.08094](#).

Further directions of study:

- ▶ Cluster algebras with anticommuting variables
- ▶ Quantization of super-Teichmüller spaces (first attempt by J. Teschner et al. [arXiv:1512.02617](#))
- ▶ Application to supermoduli theory and calculation of superstring amplitudes, which are highly nontrivial due to recent results of R. Donagi and E. Witten
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i) Superspaces and supermanifolds

Let $\Lambda(\mathbb{K}) = \Lambda^0(\mathbb{K}) \oplus \Lambda^1(\mathbb{K})$ be an exterior algebra over field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with (in)finitely many generators $1, e_1, e_2, \dots$, so that

$$a = a^\# + \sum_i a_i e_i + \sum_{ij} a_{ij} e_i \wedge e_j + \dots, \quad \# : \Lambda(\mathbb{K}) \rightarrow \mathbb{K}$$

Then superspace $\mathbb{K}^{(n|m)}$ is:

$$\mathbb{K}^{(n|m)} = \{(z_1, z_2, \dots, z_n | \theta_1, \theta_2, \dots, \theta_m) : z_i \in \Lambda^0(\mathbb{K}), \theta_j \in \Lambda^1(\mathbb{K})\}$$

One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as *even and odd coordinates*.

- Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):

$$H^+ = \{(z | \theta_1, \theta_2, \dots, \theta_N) \in \mathbb{C}^{(1|N)} \mid \operatorname{Im} z^\# > 0\}$$

- Positive superspace:

$$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \dots, z_n | \theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^{(m|n)} \mid z_i^\# > 0, i = 1, \dots, n\}$$

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ii) Supergroup $OSp(1|2)$

$(2|1) \times (2|1)$ supermatrices g , obeying the relation

$$g^{st} J g = J,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and where the supertranspose g^{st} of g is given by

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{implies} \quad g^{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

We want connected component of identity, so we assume that Berezinian (super-analogue of determinant) = 1.

Lie superalgebra: three even h, X_{\pm} and two odd generators v_{\pm} satisfying the commutation relations

$$[h, v_{\pm}] = \pm v_{\pm}, \quad [v_{\pm}, v_{\pm}] = \mp 2X_{\pm}, \quad [v_{+}, v_{-}] = h.$$

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$$z \rightarrow \frac{az + b}{cz + d} + \eta \frac{\gamma z + \delta}{(cz + d)^2},$$
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Factor H^+/Γ , where Γ is a super-Fuchsian group and H^+ is the $\mathcal{N} = 1$ super-half-plane are called super-Riemann surfaces.

We note that there are more general fractional-linear transformations acting on H^+ . They correspond to $SL(1|2)$ supergroup, and factors H^+/Γ give $(1|1)$ -supermanifolds which have relation to $\mathcal{N} = 2$ super-Teichmüller theory.

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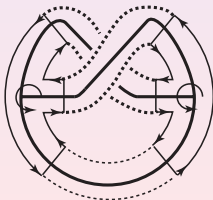
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iii) Ideal triangulations and trivalent fatgraphs

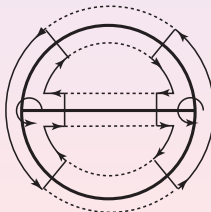
- Ideal triangulation of F : triangulation Δ of F with punctures at the vertices, so that each arc connecting punctures is not homotopic to a point rel punctures.
- Trivalent fatgraph: trivalent graph τ with cyclic orderings on half-edges about each vertex.

$\tau = \tau(\Delta)$, if the following is true:

- 1) one fatgraph vertex per triangle
- 2) one edge of fatgraph intersects one shared edge of triangulation.



Fatgraph for F_1^1



Fatgraph for F_0^3

iv) ($\mathcal{N} = 1$) Super-Teichmüller space

From now on let

$$ST(F) = \text{Hom}'(\pi_1(F), OSp(1|2))/OSp(1|2).$$

Super-Fuchsian representations comprising Hom' are defined to be those whose projections

$$\pi_1 \rightarrow OSp(1|2) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$$

are Fuchsian group, corresponding to F .

Trivial bundle $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$ is called decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space $ST(F)$ ($S\tilde{T}(F)$) has 2^{2g+s-1} connected components labeled by spin structures on F .

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are Fuchsian group, corresponding to F .

Trivial bundle $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$ is called decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space $ST(F)$ ($S\tilde{T}(F)$) has 2^{2g+s-1} connected components labeled by spin structures on F .

v) Spin structures

Let M be an oriented n -dimensional Riemannian manifold, P_{SO} is an orthonormal frame bundle, associated with TM . A spin structure is a 2-fold covering map $P \rightarrow P_{SO}$, which restricts to $Spin(n) \rightarrow SO(n)$ on each fiber.

There are several ways to describe spin structures on F :

- D. Johnson:

Quadratic forms $q : H_1(F, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$, which are quadratic for the intersection pairing $\cdot : H_1 \otimes H_1 \rightarrow \mathbb{Z}_2$, i.e. $q(a + b) = q(a) + q(b) + a \cdot b$ if $a, b \in H_1$.

- D. Cimasoni and N. Reshetikhin:

Combinatorial description of spin structures in terms of the so-called Kasteleyn orientations and dimer configurations on the one-skeleton of a suitable CW decomposition of F . They derive formula for the quadratic form in terms of that combinatorial data.

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- S. Natanzon:

A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \rightarrow SL(2, \mathbb{R})$ of $\rho : \pi_1 \rightarrow PSL_2(\mathbb{R})$. Quadratic form q is computed using the following rules: $\text{trace } \tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.

We gave another combinatorial formulation of spin structures on F (one of the main results of [arXiv:1509.06302](https://arxiv.org/abs/1509.06302)):

- Equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips:

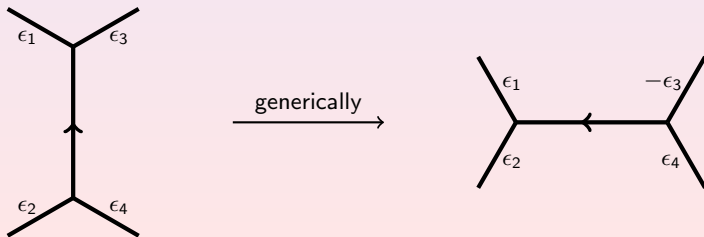


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Fix a surface $F = F_g^s$ as above and

- ▶ $\tau \subset F$ is some trivalent fatgraph spine
- ▶ ω is an orientation on the edges of τ whose class in $\mathcal{O}(\tau)$ determines the component C of $S\tilde{T}(F)$

Then there are global affine coordinates on C :

- ▶ one even coordinate called a λ -length for each edge
- ▶ one odd coordinate called a μ -invariant for each vertex of τ ,

the latter of which are taken modulo an overall change of sign.

Alternating the sign in one of the fermions corresponds to the reflection on the spin graph.

The above λ -lengths and μ -invariants establish a real-analytic homeomorphism

$$C \rightarrow \mathbb{R}_+^{6g-6+3s|4g-4+2s} / \mathbb{Z}_2.$$

Coordinates on $S\tilde{T}(F)$

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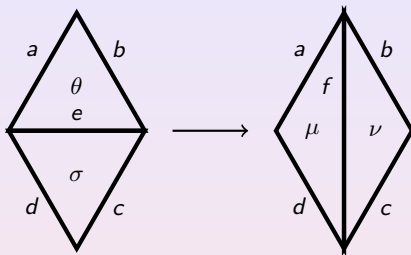
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When all a, b, c, d are different edges of the triangulations of F ,



Ptolemy transformations are as follows:

$$ef = (ac + bd) \left(1 + \frac{\sigma \theta \sqrt{\chi}}{1 + \chi} \right),$$

$$\nu = \frac{\sigma + \theta \sqrt{\chi}}{\sqrt{1 + \chi}}, \quad \mu = \frac{\sigma \sqrt{\chi} - \theta}{\sqrt{1 + \chi}}.$$

$\chi = \frac{ac}{bd}$ denotes the cross-ratio, and the evolution of spin graph follows from the construction associated to the spin graph evolution rule.

These coordinates are natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{\Gamma} \in ST(F)$, then $\tilde{\varphi}(\tilde{\Gamma})$ is determined by the orientation and coordinates on edges and vertices of $\varphi(\tau)$ induced by φ from the orientation ω , the λ -lengths and μ -invariants on τ .

There is an even 2-form on $ST(F)$ which is invariant under super Ptolemy transformations, namely,

$$\omega = \sum_v d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a - (d\theta)^2,$$

where the sum is over all vertices v of τ where the consecutive half edges incident on v in clockwise order have induced λ -lengths a, b, c and θ is the μ -invariant of v .

Coordinates on $ST(F)$:

Take instead of λ -lengths shear coordinates $z_e = \log \left(\frac{ac}{bd} \right)$ for every edge e , which are subject to linear relation: the sum of all z_e adjacent to a given vertex = 0.

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$OSp(1|2)$ acts in super-Minkowski space $\mathbb{R}^{2,1|2}$.

If $A = (x_1, x_2, y, \phi, \theta)$ and $A' = (x'_1, x'_2, y', \phi', \theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A' \rangle = \frac{1}{2}(x_1 x'_2 + x'_1 x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- ▶ Superhyperboloid \mathbb{H} consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$
- ▶ Positive super light cone L^+ consisting of points $B \in \mathbb{R}^{2,1|2}$ satisfying $\langle B, B \rangle = 0$,

where $x_1^\#, x_2^\# \geq 0$.

Equivariant projection from \mathbb{H} on the upper half plane H^+ is given by the formulas:

$$\eta = \frac{\theta}{x_2}(1 + iy) - i\phi, \quad z = \frac{i - y - i\phi\theta}{x_2}$$

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$OSp(1|2)$ does not act transitively on L^+ :

The space of orbits is labelled by odd variable up to a sign.

We pick an orbit of the vector $(1, 0, 0, 0, 0)$ and denote it L_0^+ .

The equivariant projection from L_0^+ to $\mathbb{R}^{1|1} = \partial H^+$ is given by:

$$(x_1, x_2, y, \phi, \psi) \rightarrow (z, \eta), \quad z = \frac{-y}{x_2}, \quad \eta = \frac{\psi}{x_2}, \quad \text{if } x_2^\# \neq 0.$$

Goal: Construction of the π_1 -equivariant lift for all the data from the universal cover \tilde{F} , associated to its triangulation to L_0^+ .

Such equivariant lift gives the representation of π_1 in $OSp(1|2)$.

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Orbits of 2 and 3 points in L_0^+

- There is a unique $OSp(1|2)$ -invariant of two linearly independent vectors $A, B \in L_0^+$, and it is given by the pairing $\langle A, B \rangle$, the square root of which we will call λ -length.

Let $\zeta^b \zeta^e \zeta^a$ be a positive triple in L_0^+ . Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even r, s, t , which have positive bodies, and odd θ so that

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- The moduli space of $OSp(1|2)$ -orbits of positive triples in the light cone is given by $(a, b, e, \theta) \in \mathbb{R}_+^{3|1} / \mathbb{Z}_2$, where \mathbb{Z}_2 acts by fermionic reflection.

Here λ -lengths

$$a^2 = \langle \zeta^b, \zeta^e \rangle, \quad b^2 = \langle \zeta^a, \zeta^e \rangle, \quad e^2 = \langle \zeta^a, \zeta^b \rangle.$$

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On the superline $\mathbb{R}^{1|1}$ parameter θ is known as *Manin invariant*.

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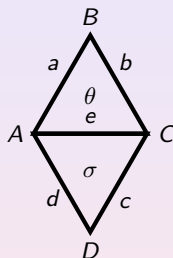
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Orbits of 4 points in L_0^+ : basic calculation

Suppose points A, B, C are put in the standard position.

The 4th point D : $(x_1, x_2, -y, \rho, \xi)$, so that two new λ -lengths are c, d .



Fixing the sign of θ , we fix the sign of Manin invariant σ as follows:

$$x_1 = \sqrt{2} \frac{cd}{e} \chi^{-1}, \quad x_2 = \sqrt{2} \frac{cd}{e} \chi, \quad \lambda = -\sqrt{2} \frac{cd}{e} \sqrt{\chi} \sigma, \quad \rho = \sqrt{2} \frac{cd}{e} \sqrt{\chi}^{-1} \sigma$$

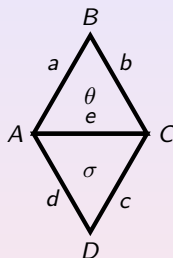
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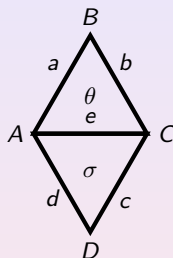
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The lift of ideal triangulation to super-Minkowski space

Penner coordinates
on super-Teichmüller
spaces

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Denote:

- ▶ Δ is ideal triangulation of F , $\tilde{\Delta}$ is ideal triangulation of the universal cover \tilde{F}
- ▶ Δ_∞ ($\tilde{\Delta}_\infty$)-collection of ideal points of F (\tilde{F}).

Consider Δ together with:

- the orientation on the fatgraph $\tau(\Delta)$,
- coordinate system $\tilde{C}(F, \Delta)$, i.e.
- positive even coordinate for every edge
- odd coordinate for every triangle

We call coordinate vectors \vec{c} , \vec{c}' equivalent if they are identical up to overall reflection of sign of odd coordinates.

Let $C(F, \Delta) \equiv \tilde{C}(F, \Delta) / \sim$. This implies that

$$C(F, \Delta) \simeq \mathbb{R}_+^{6g+3s-6|4g+2s-4} / \mathbb{Z}_2$$

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$$C(F, \Delta) \simeq \mathbb{R}_+^{6g+3s-6|4g+2s-4} / \mathbb{Z}_2$$

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Denote:

- ▶ Δ is ideal triangulation of F , $\tilde{\Delta}$ is ideal triangulation of the universal cover \tilde{F}
- ▶ Δ_∞ ($\tilde{\Delta}_\infty$)-collection of ideal points of F (\tilde{F}).

Consider Δ together with:

- the orientation on the fatgraph $\tau(\Delta)$,
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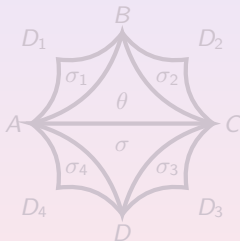
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Then there exist a lift for each $\vec{c} \in \ell : \tilde{\Delta}_\infty \rightarrow L_0^+$, with the property:

for every quadrilateral $ABCD$, if the arrow is pointing from σ to θ then the lift is given by the picture from the previous slide up to post-composition with the element of $OSP(1|2)$.

The construction of ℓ can be done in a recursive way:

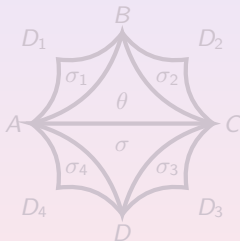


Such lift is unique up to post-composition with $OSP(1|2)$ group element and it is π_1 -equivariant. This allows us to construct representation of π_1 in $OSP(1|2)$, based on the provided data.

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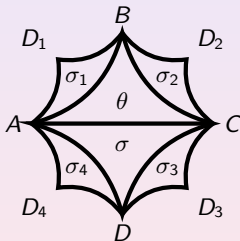


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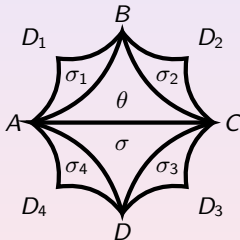


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Theorem

Fix $F, \Delta, \tau(\Delta)$ as before. Let ω be an orientation, corresponding to a specified spin structure s of F . Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

$$\ell_\omega : \tilde{\Delta}_\infty \rightarrow L_0^+$$

which is uniquely determined up to post-composition by $OSp(1|2)$ under admissibility conditions discussed above, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates.

There is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \rightarrow OSp(1|2)$, uniquely determined up to conjugacy by an element of $OSp(1|2)$ such that

- (1) ℓ is π_1 -equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;
- (2) $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

$$\rho : \pi_1 \xrightarrow{\hat{\rho}} OSp(1|2) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$$

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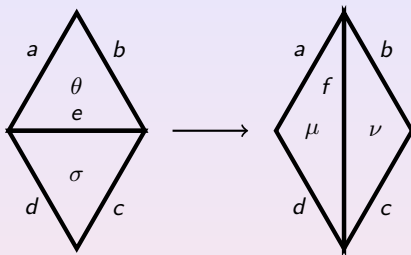
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The super-Ptolemy transformations



$$ef = (ac + bd) \left(1 + \frac{\sigma \theta \sqrt{\chi}}{1 + \chi} \right),$$

$$\nu = \frac{\sigma + \theta \sqrt{\chi}}{\sqrt{1 + \chi}}, \quad \mu = \frac{\sigma \sqrt{\chi} - \theta}{\sqrt{1 + \chi}}$$

are the consequence of light cone geometry.

The space of all such lifts ℓ_ω coincides with the decorated super-Teichmüller space $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$.

In order to remove the decoration, one can pass to shear coordinates $z_e = \log \left(\frac{ac}{bd} \right)$.

It is easy to check that the 2-form

$$\omega = \sum_{\Delta} d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a - (d\theta)^2,$$

is invariant under the flip transformations. This is a generalization of the formula for Weil-Petersson 2-form.

$\mathcal{N} = 2$ super-Teichmüller theory: prerequisites

$\mathcal{N} = 2$ super-Teichmüller space is related to $OSP(2|2)$ supergroup of rank 2.

It is more useful to work with its 3×3 incarnation, which is isomorphic to $\Psi \ltimes SL(1|2)_0$, where Ψ is a certain automorphism of the Lie algebra $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$.

$SL(1|2)_0$ is a supergroup, consisting of supermatrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix}$$

such that $f > 0$ and their Berezinian = 1.

This group acts on the space $\mathbb{C}^{1|2}$ as superconformal fractional-linear transformations.

As before, $\mathcal{N} = 2$ super-Fuchsian groups are the ones whose projections

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Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2, \mathbb{R})$.

Therefore, the construction of coordinates requires a new notion:
 \mathbb{R}_+ -graph connection.

A G -graph connection on τ is the assignment $h_e \in G$ to each oriented edge e of τ so that $h_{\bar{e}} = h_e^{-1}$ if \bar{e} is the opposite orientation to e .

Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex v of τ such that $h'_e = t_v h_e t_w^{-1}$ for each oriented edge $e \in \tau$ with initial point v and terminal point w .

The moduli space of flat G -connections on F is isomorphic to the space of equivalent G -graph connections on τ .

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The data giving the coordinate system $\tilde{C}(F, \Delta)$ is as follows:

- ▶ we assign to each edge of Δ a positive even coordinate e ;
- ▶ we assign to each triangle of Δ two odd coordinates (θ_1, θ_2) ;
- ▶ we assign to each edge e of a triangle of Δ a positive even coordinate h_e , called the *ratio*, such that if h_e and h'_e are assigned to two triangles sharing the same edge e , they satisfy $h_e h'_e = 1$.

The odd coordinates are defined up to overall sign changes $\theta_i \rightarrow -\theta_i$, as well as an overall involution $(\theta_1, \theta_2) \rightarrow (\theta_2, \theta_1)$.

Assignment implies that the ratios $\{h_e\}$ uniquely define an \mathbb{R}_+ -graph connection on $\tau(\Delta)$.

Gauge transformations: if h_a, h_b, h_e are ratios assigned to a triangle T with odd coordinate (θ_1, θ_2) , then a *vertex rescaling at T* is the following transformation:

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Gauge transformations: if h_a, h_b, h_e are ratios assigned to a triangle T with odd coordinate (θ_1, θ_2) , then a *vertex rescaling at T* is the following transformation:

$$(h_a, h_b, h_e, \theta_1, \theta_2) \rightarrow (uh_a, uh_b, uh_e, u^{-1}\theta_1, u\theta_2)$$

for some $u > 0$.

The data giving the coordinate system $\tilde{C}(F, \Delta)$ is as follows:

- ▶ we assign to each edge of Δ a positive even coordinate e ;
- ▶ we assign to each triangle of Δ two odd coordinates (θ_1, θ_2) ;
- ▶ we assign to each edge e of a triangle of Δ a positive even coordinate h_e , called the *ratio*, such that if h_e and h'_e are assigned to two triangles sharing the same edge e , they satisfy $h_e h'_e = 1$.

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We say that two coordinate vectors of $\tilde{C}(F, \Delta)$ are equivalent if they are related by a finite number of such vertex rescalings (i.e. gauge transformations). In particular the underlying \mathbb{R}_+ -graph connections on τ are equivalent.

Let $C(F, \Delta) := \tilde{C}(F, \Delta) / \sim$ be the equivalent classes of coordinate vectors. Then it can be represented by coordinates with $h_a h_b h_e = 1$ for the ratios of the same triangle. This implies that

$$C(F, \Delta) \simeq \mathbb{R}_+^{8g+4s-7|8g+4s-8} / \mathbb{Z}_2 \times \mathbb{Z}_2$$

Note, that two involutions we have, one corresponding to the fermion reflection and another one corresponding to Ψ give rise to two spin structures, which enumerate components of the $\mathcal{N} = 2$ super-Teichmüller space.

The light cone L_0^+ and upper sheet hyperboloid \mathbb{H}_0^+ in this case are certain orbits in a pseudo-euclidean superspace $\mathbb{R}^{2,2|4}$.

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Theorem

Fix F, Δ, τ as before. Let $\omega_{sign} := \omega_{s_{sign}, \tau}$ be a representative, corresponding to a specified spin structure s_{sign} of F , and let $\omega_{inv} := \omega_{s_{inv}, \tau}$ be the representative of another spin structure s_{inv} .

Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$ there exists a map called the lift,

$$\ell_{\omega_{sign}, \omega_{inv}} : \tilde{\Delta}_{\infty} \rightarrow L_0^+,$$

which is uniquely determined up to post-composition by $OSp(2|2)$ under some admissibility conditions, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates. Then there is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \rightarrow OSp(2|2)$, uniquely determined up to conjugacy by an element of $OSp(2|2)$ such that

- (1) ℓ is π_1 -equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_{\infty}$;
- (2) $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

$$\rho : \pi_1 \xrightarrow{\hat{\rho}} OSp(2|2) \rightarrow SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$$

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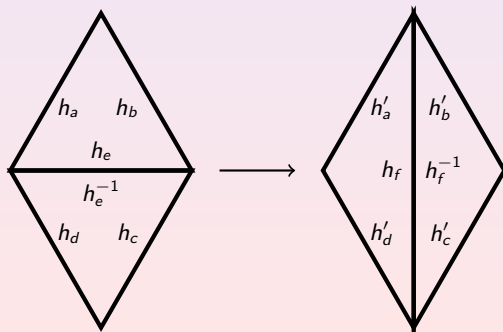
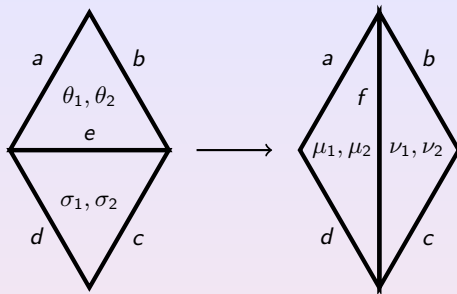
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Generic Ptolemy transformations are:



and the transformation formulas are as follows:

$$ef = (ac + bd) \left(1 + \frac{h_e^{-1} \sigma_1 \theta_2}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})} + \frac{h_e \sigma_2 \theta_1}{2(\sqrt{\chi} + \sqrt{\chi}^{-1})} \right),$$

$$\mu_1 = \frac{h_e \theta_1 + \sqrt{\chi} \sigma_1}{\mathcal{D}}, \quad \mu_2 = \frac{h_e^{-1} \theta_2 + \sqrt{\chi} \sigma_2}{\mathcal{D}},$$

$$\nu_1 = \frac{\sigma_1 - \sqrt{\chi} h_e \theta_1}{\mathcal{D}}, \quad \nu_2 = \frac{\sigma_2 - \sqrt{\chi} h_e^{-1} \theta_2}{\mathcal{D}},$$

$$h'_a = \frac{h_a}{h_e c_\theta}, \quad h'_b = \frac{h_b c_\theta}{h_e}, \quad h'_c = h_c \frac{c_\theta}{c_\mu}, \quad h'_d = h_d \frac{c_\nu}{c_\theta}, \quad h_f = \frac{c_\sigma}{c_\theta^2},$$

where

$$\mathcal{D} := \sqrt{1 + \chi + \frac{\sqrt{\chi}}{2} (h_e^{-1} \sigma_1 \theta_2 + h_e \sigma_2 \theta_1)},$$

$$c_\theta := 1 + \frac{\theta_1 \theta_2}{6}.$$

The space of all lifts $\ell_{\omega_{sign}, \omega_{inv}}$ is called decorated $\mathcal{N} = 2$ super-Teichmüller space, which is again \mathbb{R}_+^s -bundle over $\mathcal{N} = 2$ super-Teichmüller space.

Removal of the decoration is done using a similar procedure, using shear coordinates.

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- 1) Cluster superalgebras
- 2) Weil-Petersson-form in $\mathcal{N} = 2$ case
- 3) Duality between $\mathcal{N} = 2$ super Riemann surfaces and $(1|1)$ -supermanifolds
- 4) Quantization of super-Teichmüller spaces
- 5) Weil-Petersson volumes
- 6) Application to supermoduli theory and calculation of superstring amplitudes
- 7) Higher super-Teichmüller theory for supergroups of higher rank

Thank you!