# Einstein field equations, Courant algebroids and Homotopy algebras

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# Outline

Sigma-models and conformal invariance conditions

Beltrami-Courant differential and first order sigma-models

Vertex/Courant algebroids, $G_\infty$-algebras and quasiclassical limit

Einstein Equations from $G_\infty$-algebras

# Sigma-models and conformal invariance conditions

Sigma-models for string theory in curved spacetimes:

Let $X : \Sigma \to M$, where $\Sigma$ is a compact Riemann surface (worldsheet) and $M$ is a Riemannian manifold (target space).

Action functional of sigma model:

$$S_{so} = \frac{1}{4\pi h}\int_{\Sigma}(G_{\mu\nu}(X)dX^{\mu} \wedge *dX^{\nu} + X^{*}B)$$

where $G$ is a metric on $M$, $B$ is a 2-form on $M$.

Symmetries:

i) conformal symmetry on the worldsheet,

ii) diffeomorphism symmetry and $B \to B + d\lambda$ on target space.

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On the quantum level one can add one more term to the action (due to E. Fradkin and A. Tseytlin):

$$S_{so} \to S_{so}^{\Phi} = S_{so} + \int_{\Sigma} \Phi(X) R^{(2)}(\gamma)\mathrm{vol}_{\Sigma},$$

where function $\Phi$ is called *dilaton*, $\gamma$ is a metric on $\Sigma$.

In order to make sense of path integral

$$Z = \int DX \quad e^{-S_{so}^{\Phi}(X,\gamma)}$$

one has to apply renormalization procedure, so that $G$, $B$, $\Phi$ depend on certain *cutoff* parameter $\mu$, so that in general quantum theory is not conformally invariant.

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Conformal invariance conditions

$$\mu \frac{d}{d\mu} G_{\mu\nu} = \beta^G_{\mu\nu}(G, B, \Phi, h) = 0, \quad \mu \frac{d}{d\mu} B_{\mu\nu} = \beta^B_{\mu\nu}(G, B, \Phi, h) = 0,$$
$$\mu \frac{d}{d\mu} \Phi = \beta^\Phi(G, B, \Phi, h) = 0$$

at the level $h^0$ turn out to be Einstein Equations with 2-form field $B$ and dilaton $\Phi$:

$$R_{\mu\nu} = \frac{1}{4} H_\mu^{\lambda\rho} H_{\nu\lambda\rho} - 2\nabla_\mu \nabla_\nu \Phi,$$
$$\nabla^\mu H_{\mu\nu\rho} - 2(\nabla^\lambda \Phi) H_{\lambda\nu\rho} = 0,$$
$$4(\nabla_\mu \Phi)^2 - 4\nabla_\mu \nabla^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0,$$

where 3-form $H = dB$, and $R_{\mu\nu}$, $R$ are Ricci and scalar curvature correspondingly.

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In the early days of string theory:

Linearized Einstein Equations and their symmetries:
$(G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu},\ B_{\mu\nu} = b_{\mu\nu},\ \Phi = \phi)$:

$$Q^\eta \Psi(s, b, \phi) = 0, \quad \Psi^s(s, b, \phi) \to \Psi(s, b, \phi) + Q^\eta \Lambda$$

in a semi-infinite complex associated to Virasoro module of Hilbert space of states for the "free" theory, asssociated to flat metric.

It was conjectured (A. Sen, B. Zwiebach,...) in the early 90s that Einstein equations with $h$-corrections are Generalized Maurer-Cartan (GMC) Equations:

$$Q^\eta \Psi + \frac{1}{2}[\Psi, \Psi]_h + \frac{1}{3!}[\Psi, \Psi, \Psi]_h + \ldots = 0$$

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In this talk:

i) Introducing complex structure:

Proper chiral "free action" $\to$ sheaves of vertex algebras/vertex algebroids.

Metric, $B$-field $\to$ Beltrami-Courant differential.

ii) Vertex algebroids $\to$ $G_\infty$-algebras ($G$ stands for Gerstenhaber).

Quasiclassical limit:
vertex algebroid $\to$ Courant algebroid, $G_\infty$ algebra is truncated.

iii) Einstein equations and their $h$-corrections via Generalized Maurer-Cartan equation for $L_\infty$-subalgebra of $G_\infty \otimes \bar{G}_\infty$.

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# First order version of sigma-model action

We start from the action functional:

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma \mathcal{L}_0, \quad \mathcal{L}_0 = \langle p \wedge \bar{\partial} X\rangle - \langle \bar{p} \wedge \partial X\rangle,$$

where $p$, $\bar{p}$ are sections of $X^*(\Omega^{(1,0)}(M)) \otimes \Omega^{(1,0)}(\Sigma)$, $X^*(\Omega^{(0,1)}(M)) \otimes \Omega^{(0,1)}(\Sigma)$ correspondingly.

Infinitesimal local symmetries:

$$\mathcal{L}_0 \to \mathcal{L}_0 + d\xi$$

For holomorphic transformations we have:

$$X^i \to X^i - v^i(X), X^{\bar{i}} \to X^{\bar{i}} - v^{\bar{i}}(\bar{X}),$$
$$p_i \to p_i + \partial_i v^k p_k, \quad p_{\bar{i}} \to p_{\bar{i}} + \partial_{\bar{i}} v^{\bar{k}} p_{\bar{k}}$$
$$p_i \to p_i - \partial X^k(\partial_k \omega_i - \partial_i \omega_k), \quad p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}} \omega_{\bar{i}} - \partial_{\bar{i}} \omega_{\bar{k}}).$$

Not invariant under general diffeomorphisms, i.e.

$$\delta\mathcal{L}_0 = -\langle \bar{\partial} v, p \wedge \bar{\partial} X\rangle + \langle \partial \bar{v}, \bar{p} \wedge \partial X\rangle.$$

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$$\delta\mathcal{L}_0 = -\langle\bar\partial v, p\wedge\bar\partial X\rangle + \langle\partial\bar v, \bar p\wedge\partial X\rangle.$$

It is necessary to add extra terms:

$$\delta\mathcal{L}_\mu = -\langle \mu, p \wedge \bar\partial X\rangle - \langle \bar\mu, \partial X \wedge \bar p\rangle,$$

where $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}(M))$, $\bar\mu \in \Gamma(T^{(0,1)}M \otimes T^{*(1,0)}(M))$, so that: $\mu \to \mu - \bar\partial v + \dots$, $\bar\mu \to \bar\mu - \partial \bar v + \dots$.

Continuing the procedure:

$$\tilde{\mathcal{L}} = \langle p \wedge \bar\partial X\rangle - \langle \bar p \wedge \partial X\rangle - \langle \mu, p \wedge \bar\partial X\rangle - \langle \bar\mu, \partial X \wedge \bar p\rangle - \langle b, \partial X \wedge \bar\partial X\rangle,$$

where

$$\begin{aligned}
&\mu^i_{\bar j} \to \\
&\mu^i_{\bar j} - \partial_{\bar j} v^i + v^k\partial_k \mu^i_{\bar j} + v^{\bar k}\partial_{\bar k}\mu^i_{\bar j} + \mu^i_{\bar k}\partial_{\bar j} v^{\bar k} - \mu^k_{\bar j}\partial_k v^i + \mu^i_{\bar l}\mu^k_{\bar j}\partial_k v^{\bar l},\\
&b_{i\bar j} \to \\
&b_{i\bar j} + v^k\partial_k b_{i\bar j} + v^{\bar k}\partial_{\bar k} b_{i\bar j} + b_{i\bar k}\partial_{\bar j} v^{\bar k} + b_{l\bar j}\partial_i v^l + b_{i\bar k}\mu^k_{\bar j}\partial_k v^{\bar k} + b_{l\bar j}\bar\mu^{\bar k}_i\partial_{\bar k} v^l,
\end{aligned}$$

so that the transformations of $X$- and $p$- fields are:

$$\begin{aligned}
&X^i \to X^i - v^i(X,\bar X), \quad p_i \to p_i + p_k\partial_i v^k - p_k\mu^k_{\bar l}\partial_i v^{\bar l} - b_{j\bar k}\partial_i v^{\bar k}\partial X^j,\\
&X^{\bar i} \to X^{\bar i} - v^{\bar i}(X,\bar X), \quad \bar p_{\bar i} \to \bar p_{\bar i} + \bar p_{\bar k}\partial_{\bar i} v^{\bar k} - \bar p_{\bar k}\bar\mu^{\bar k}_l\partial_i v^l - b_{\bar j k}\partial_{\bar i} v^k\bar\partial X^{\bar j}.
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$$\mu^i_{\bar{j}} \to \\ \mu^i_{\bar{j}} - \partial_{\bar{j}} v^i + v^k \partial_k \mu^i_{\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} \mu^i_{\bar{j}} + \mu^i_{\bar{k}} \partial_{\bar{j}} v^{\bar{k}} - \mu^k_{\bar{j}} \partial_k v^i + \mu^i_{\bar{l}} \mu^k_{\bar{j}} \partial_k v^{\bar{l}}, \\ b_{i\bar{j}} \to \\ b_{i\bar{j}} + v^k \partial_k b_{i\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} b_{i\bar{j}} + b_{i\bar{k}} \partial_{\bar{j}} v^{\bar{k}} + b_{l\bar{j}} \partial_i v^l + b_{i\bar{k}} \mu^k_{\bar{j}} \partial_k v^{\bar{k}} + b_{l\bar{j}} \bar{\mu}^{\bar{k}}_i \partial_{\bar{k}} v^l,$$

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$$X^i \to X^i - v^i(X, \bar{X}), \quad p_i \to p_i + p_k \partial_i v^k - p_k \mu^k_{\bar{l}} \partial_i v^{\bar{l}} - b_{j\bar{k}} \partial_i v^{\bar{k}} \partial X^j, \\ X^{\bar{i}} \to X^{\bar{i}} - v^{\bar{i}}(X, \bar{X}), \quad \bar{p}_{\bar{i}} \to \bar{p}_{\bar{i}} + \bar{p}_{\bar{k}} \partial_{\bar{i}} v^{\bar{k}} - \bar{p}_{\bar{k}} \bar{\mu}^{\bar{k}}_l \partial_i v^l - b_{\bar{j}k} \partial_{\bar{i}} v^k \bar{\partial} X^{\bar{j}}.$$

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where

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$$b_{i\bar j}\to$$
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$$X^{\bar i}\to X^{\bar i} - v^{\bar i}(X,\bar X),\quad \bar p_{\bar i}\to \bar p_{\bar i} + \bar p_{\bar k}\partial_{\bar i}v^{\bar k} - \bar p_{\bar k}\bar\mu^{\bar k}_l\partial_i v^l - b_{\bar j k}\partial_{\bar i}v^k\bar\partial X^{\bar j}.$$

Similarly, for the 1-form transformation we obtain:

$$
\begin{aligned}
&b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu^i_{\bar{j}}(\partial_i\omega_k - \partial_k\omega_i) + \\
&\bar{\mu}^{\bar{s}}_i(\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}^{\bar{i}}_j\mu^s_{\bar{k}}(\partial_s\omega_{\bar{i}} - \partial_{\bar{i}}\omega_s)
\end{aligned}
$$

and

$$
\begin{aligned}
&p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_k\partial_i\omega_{\bar{s}}\partial X^k, \\
&p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial} X^r - \mu^s_{\bar{k}}\partial_i\omega_s\bar{\partial} X^{\bar{k}}.
\end{aligned}
$$

For simplicity:

$$
\begin{aligned}
&E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}}, \\
&\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.
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$$\bar{\mu}^{\bar{s}}_{i}(\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}^{\bar{i}}_{j}\mu^{s}_{\bar{k}}(\partial_s\omega_{\bar{i}} - \partial_{\bar{i}}\omega_s)$$

and

$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_{k}\partial_i\omega_{\bar{s}}\partial X^k,$$
$$p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial} X^r - \mu^{s}_{\bar{k}}\partial_i\omega_s\bar{\partial} X^{\bar{k}}.$$

For simplicity:

$$E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}},$$
$$\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.$$

Let $\tilde{\mathbb{M}} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$\tilde{\mathbb{M}} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$D\alpha = \left( \begin{array}{cc} 0 & \bar{\partial} v \\ \partial \bar{v} & \partial \bar{\omega} - \bar{\partial} \omega \end{array} \right).$$

Then the transformation of $\tilde{\mathbb{M}}$ is:

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Let us describe $\phi_1, \phi_2$ algebraically. In order to do that we need to pass to jet bundles, i.e.

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One can write formally:

$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$
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where $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$, $f^I \in J^\infty(\mathcal{O}_M)$ and $\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$, $\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)$. Then

$$\phi_1(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where $[\cdot,\cdot]_D$ is a *Dorfman bracket*:

$$[v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v \omega,$$
$$[\omega, v]_D = -i_v d\omega, \quad [\omega_1, \omega_2]_D = 0.$$

Courant bracket is the antisymmetrized version of $[\cdot,\cdot]_D$.
Similarly:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}}$$
$$\frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J (\bar{f}^I) \bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J (f^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

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The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on $M$) via sheaves of VOA on $M$ (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set $U$ of $M$ we have VOA:

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generated by:

$$[X^i(z), p_j(w)] = h\delta^i_j \delta(z-w), \quad i,j = 1,2,\dots,D/2$$
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# Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on $M$) via sheaves of VOA on $M$ (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).
On the open set $U$ of $M$ we have VOA:

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The Virasoro element is:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{h} : \langle p(z)\partial X(z)\rangle : + \partial^2\phi'(X(z)).$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}(n^3 - n)\delta_{n,-m}$$

corresponding to correction:

$$\mathcal{L}_0 \to \mathcal{L}_{\phi'} = \langle p \wedge \bar{\partial} X\rangle - 2\pi i h R^{(2)}(\gamma)\phi'(X)$$

where $\phi' = \log\Omega$, where $\Omega(X)dX^1 \wedge \cdots \wedge dX^n$ is a holomorphic volume form, i.e. for globally defined $T(z)$, $M$ has to be Calabi-Yau.
The space $V$ is a lowest weight module for the above Virasoro algebra.

$V$ can be reproduced from $V_0$ and $V_1$ as a *vertex envelope*. The structure of vertex algebra imposes algebraic relations on $V_0 \oplus V_1$ giving it a structure of a *vertex algebroid*.

In our case: $V_0 \to \mathcal{O}^h_U = \mathcal{O}_U \otimes \mathbb{C}[h, h^{-1}]$,
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The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1} = c(z)T(z) + :c(z)\partial c(z) b(z):$$

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# The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

$$(A, B)_h = Res_z \frac{A(z)B}{z}$$

$$Q(a_1, a_2)_h = (Qa_1, a_2)_h + (-1)^{|a_1|}(a_1, Qa_2)_h,$$
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\mathcal{O}_M^h & \xleftarrow{id} & \mathcal{O}_M^h & & \mathcal{O}_M^h & \xleftarrow{-id} & \mathcal{O}_M^h
\end{array}$$

# The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

$$(A,B)_h = Res_z \frac{A(z)B}{z}$$

$$\begin{aligned}
&Q(a_1,a_2)_h = (Qa_1,a_2)_h + (-1)^{|a_1|}(a_1,Qa_2)_h,\\
&(a_1,a_2)_h - (-1)^{|a_1||a_2|}(a_2,a_1)_h =\\
&Qm(a_1,a_2) + m(Qa_1,a_2) + (-1)^{|a_1|}m(a_1,Qa_2),\\
&Q(a_1,a_2,a_3)_h + (Qa_1,a_2,a_3)_h + (-1)^{|a_1|}(a_1,Qa_2,a_3)_h +\\
&(-1)^{|a_1|+|a_2|}(a_1,a_2,Qa_3)_h = ((a_1,a_2)_h,a_3)_h - (a_1,(a_2,a_3)_h)_h
\end{aligned}$$

Operator **b** of degree -1 (0-mode of $b(z)$) on $(\mathcal{F}^{\cdot}_h, Q)$ which anticommutes with $Q$:

$$\begin{array}{ccc}
 & \mathcal{V}^h \xleftarrow{-id} \mathcal{V}^h & \\
 & \bigoplus \quad\quad \bigoplus & \\
\mathcal{O}^h_M \xleftarrow{id} \mathcal{O}^h_M & & \mathcal{O}^h_M \xleftarrow{-id} \mathcal{O}^h_M
\end{array}$$

One can define a bracket:

$$(-1)^{|a_1|}\{a_1, a_2\}_h = \mathbf{b}(a_1, a_2)_h - (\mathbf{b}a_1, a_2)_h - (-1)^{|a_1|}(a_1\mathbf{b}a_2)_h,$$

so that together with $Q$, $(\cdot,\cdot)_h$ it satisfies the relations of homotopy Gerstenhaber algebra:

$$\{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\}_h =$$
$$(-1)^{|a_1|-1}(Qm'_h(a_1, a_2) - m'_h(Qa_1, a_2) - (-1)^{|a_2|}m'_h(a_1, Qa_2)),$$
$$\{a_1, (a_2, a_3)_h\}_h = (\{a_1, a_2\}_h, a_3)_h + (-1)^{(|a_1|-1)||a_2|}(a_2, \{a_1, a_3\}_h)_h,$$
$$\{(a_1, a_2)_h, a_3\}_h - (a_1, \{a_2, a_3\}_h)_h - (-1)^{(|a_3|-1)|a_2|}(\{a_1, a_3\}_h, a_2)_h =$$
$$(-1)^{|a_1|+|a_2|-1}(Qn'_h(a_1, a_2, a_3) - n'_h(Qa_1, a_2, a_3) -$$
$$(-1)^{|a_1|}n'_h(a_1, Qa_2, a_3) - (-1)^{|a_1|+|a_2|}n'_h(a_1, a_2, Qa_3),$$
$$\{\{a_1, a_2\}_h, a_3\}_h - \{a_1, \{a_2, a_3\}_h\}_h +$$
$$(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, \{a_1, a_3\}_h\}_h = 0.$$

The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to $G_\infty$-algebra.

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# Homotopy algebras: $G_\infty$, $L_\infty$, $C_\infty$

Let $A$ be a graded vector space, consider free graded Lie algebra $Lie(A)$.

$$Lie^{k+1}(A) = [A, Lie^k A], \quad Lie^1(A) = A.$$

Consider free graded commutative algebra $GA$ on the suspension $(Lie(A))[-1]$, i.e.

$$GA = \oplus_n \bigwedge^n Lie(A)[-n]$$

There are natural $[\cdot,\cdot]$, $\wedge$ operations on $GA$ of degree -1, 0 correpondingly, generating a Gerstenhaber algebra.

A $G_\infty$-*algebra* (Tamarkin, Tsygan, 2000) is a graded space $V$ with a differential $\partial$ of degree 1 of $G(V[1]^*)$, such that $\partial$ is a derivation w.r.t bracket and the product.

Multiplicative Ideals, preserved by $\partial$: $I_1$-generated by the commutant of $Lie(V[1]^*)$, $I_2 = \bigwedge_{n\geq 2}(Lie(V[1]^*)[-n]$. That induces differentials on corresponding factors: $\bigwedge_{n\geq 1}(V[1]^*)[-n]$ and $Lie(V[1]^*)[-1]$. The resulting structures on $V$ are called $L_\infty$-*algebra* and $C_\infty$-*algebra* correspondingly.

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$$V[1]^* \to Lie^{k_1}(V[1]^*) \wedge \cdots \wedge Lie^{k_n}(V[1]^*).$$

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of degree $3 - n - k_1 - \ldots - k_n$, satisfying bilinear relations.

In our previous notation $m_1 = Q$, $m_2$-symmetrized LZ product, $m_{1,1}$–antisymmetrized LZ bracket.

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An important feature of $L_\infty$ algebra is a Maurer-Cartan equation ($\Phi$ is of degree 2) :

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# Quasiclassical limit of LZ $G_\infty$ algebra

The following complex $(\mathcal{F}^{\cdot}, Q)$:

$$\begin{tikzcd}
 & \mathcal{V} \arrow[ddr, "-\frac{1}{2}h\mathrm{div}"] & h\mathcal{V} \arrow[ddr, "\frac{1}{2}h\mathrm{div}"] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{O}_M \arrow[uur, "\partial"] & h\mathcal{O}_M \arrow[uur, "\partial"] \arrow[r, "id"'] & h\mathcal{O}_M & h^2\mathcal{O}_M
\end{tikzcd}$$

is a subcomplex of $(\mathcal{F}_h^{\cdot}, Q)$. Then

$$(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\}_h : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h],$$
$$\mathbf{b} : \mathcal{F}^i \to h\mathcal{F}^{i-1}[h],$$

so that

$$(\cdot,\cdot)_0 = \lim_{h\to 0}(\cdot,\cdot)_h, \quad \{\cdot,\cdot\}_0 = \lim_{h\to 0} h^{-1}\{\cdot,\cdot\}_h, \quad \mathbf{b}_0 = \lim_{h\to 0} h^{-1}\mathbf{b}$$

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is a subcomplex of $(\mathcal{F}_h^{\cdot}, Q)$.Then

$$
\begin{aligned}
&(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\}_h : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h], \\
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$$

so that

$$
(\cdot,\cdot)_0 = \lim_{h\to 0}(\cdot,\cdot)_h, \quad \{\cdot,\cdot\}_0 = \lim_{h\to 0} h^{-1}\{\cdot,\cdot\}_h, \quad \mathbf{b}_0 = \lim_{h\to 0} h^{-1}\mathbf{b}
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are well defined.

# Quasiclassical limit of LZ $G_\infty$ algebra

The following complex $(\mathcal{F}^{\cdot}, Q)$:

$$\begin{tikzcd}
 & \mathcal{V} \arrow[dr, "-\frac{1}{2}h\mathrm{div}"] & h\mathcal{V} \arrow[dr, "\frac{1}{2}h\mathrm{div}"] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{O}_M \arrow[uur, "\partial"] & h\mathcal{O}_M \arrow[uur, "\partial"] \arrow[r, "id"'] & h\mathcal{O}_M & h^2\mathcal{O}_M
\end{tikzcd}$$

is a subcomplex of $(\mathcal{F}_h^{\cdot}, Q)$.Then

$$(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\}_h : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h],$$
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are well defined.

The symmetrized operations $(\cdot,\cdot)_0$, $\{\cdot,\cdot\}_0, \dots$ satisfy the relations of the homotopy Gerstenhaber algebra, so that all non-covariant higher-order terms disappear from the multilinear operations.

The resulting $C_\infty$ and $L_\infty$ algebras are reduced to $C_3$ and $L_3$ algebras.

A.Z., Comm. Math. Phys. 303 (2011) 331-359.

Conjecture: This $G_\infty$-algebra is the $G_3$-algebra (no homotopies beyond trilinear operations).

Classical limit procedure for vertex algebroid (due to P. Bressler):
$[\cdot,\cdot]_0 = \lim_{h\to 0}\frac{1}{h}[\cdot,\cdot]$, $\pi_0 = \lim_{h\to 0}\frac{1}{h}\pi$, $\langle\cdot,\cdot\rangle_0 = \lim_{h\to 0}\frac{1}{h}\langle\cdot,\cdot\rangle$.

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A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with a structure of a Leibniz $\mathbb{C}$-algebra $[\cdot,\cdot]_0 : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \to \mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0 : \mathcal{Q} \to \Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle \cdot,\cdot \rangle : \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{Q} \to \mathcal{O}_M$, a derivation $\partial : \mathcal{O}_M \to \mathcal{Q}$ which satisfy

$$\pi \circ \partial = 0, \quad [q_1, fq_2]_0 = f[q_1,q_2]_0 + \pi_0(q_1)(f)q_2$$
$$\langle [q,q_1],q_2\rangle + \langle q_1,[q,q_2]\rangle = \pi_0(q)(\langle q_1,q_2\rangle_0),$$
$$[q,\partial(f)]_0 = \partial(\pi_0(q)(f))$$
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for $f \in \mathcal{O}_M$ and $q, q_1, q_2 \in \mathcal{Q}$.

First it was obtained as an analogue of Manin's double for Lie bialgebroid by Z-J. Liu, A. Weinsten, P. Xu.

In our case $\mathcal{Q} \cong \mathcal{O}(\mathcal{E})$, $\pi_0$ is just a projection on $\mathcal{O}(TM)$

$$[q_1,q_2]_0 = -[q_1,q_2]_D, \quad \langle q_1,q_2\rangle_0 = -\langle q_1,q_2\rangle^s, \quad \partial = d.$$

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We show that it is a part of a more general structure, homotopy Gerstenhaber algebra.

Question: Is there a direct path (avoiding vertex algebra) from Courant algebroid to $G_3$-algebra? Odd analogue of Manin double?

Remark. $C_3$-algebra is related to gauge theory. The appropriate "metric" deformation gives a Yang-Mills $C_3$-algebra on a flat space.

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# Simplest version: $G_\infty \to$ Gerstenhaber algebra

Subcomplex $(\mathcal{F}^{\cdot}_{sm}, Q)$:

$$\begin{tikzcd}
 & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "-\frac{1}{2}\mathrm{div}"] & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "\frac{1}{2}\mathrm{div}"] & \\
 & \bigoplus & \bigoplus & \\
\mathbb{C} \arrow[uur, "0"] & \mathbb{C} \arrow[uur, "0"'] \arrow[r, "i"'] & \mathcal{O}_M & \mathcal{O}_M
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The $G_\infty$ algebra degenerates to G-algebra. Moreover, due to $\mathbf{b}_0$ it is a BV-algebra. Combine chiral and antichiral part:

$$\mathbf{F}^{\cdot}_{sm} = \mathcal{F}^{\cdot}_{sm} \otimes \bar{\mathcal{F}}^{\cdot}_{sm}$$

$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^-(a_1, a_2) - (\mathbf{b}^- a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^- a_2),$$

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Maurer-Cartan elements, closed under $\mathbf{b}^-$:

$$\Gamma(T^{(1,0)}(M) \otimes T^{(0,1)}(M)) \oplus \mathcal{O}(T^{(0,1)}(M) \oplus \mathcal{O}(T^{(1,0)}(M) \oplus \mathcal{O}_M \oplus \bar{\mathcal{O}}_M$$

Components:$(g, \bar{v}, v, \phi, \bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

1). Vector field $div_\Omega g$, where $\log \Omega = -2\Phi_0 = -2(\phi' + \bar{\phi}' + \phi + \bar{\phi})$ and $\partial_i \partial_{\bar{j}} \Phi_0 = 0$, is such that its $\Gamma(T^{(1,0)}M)$, $\Gamma(T^{(0,1)}M)$ components are correspondingly holomorphic and antiholomorphic.
2). Bivector field $g \in \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$ obeys the following equation:

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These are Einstein equations with the following constraints:

$$G_{i\bar{k}} = g_{i\bar{k}}, \quad B_{i\bar{k}} = -g_{i\bar{k}}, \quad \Phi = \log\sqrt{g} + \Phi_0,$$
$$G_{ik} = G_{\bar{i}\bar{k}} = G_{ik} = G_{\bar{i}\bar{k}} = 0,$$

Physically:

$$\int [dp][d\bar{p}][dX][d\bar{X}]e^{-\frac{1}{2\pi i h}\int_\Sigma(\langle p\wedge\bar{\partial}X\rangle - \langle\bar{p}\wedge\partial X\rangle - \langle g, p\wedge\bar{p}\rangle) + \int_\Sigma R^{(2)}(\gamma)\Phi_0(X)} =$$
$$\int [dX][d\bar{X}]e^{\frac{-1}{4\pi h}\int d^2z(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu + \int R^{(2)}(\gamma)(\Phi_0(X)+\log\sqrt{g})}$$

based on computations of
A. Tseytlin and A. Schwarz, Nucl.Phys. B399 (1993) 691-708.

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$$G_{ik} = G_{\bar{i}\bar{k}} = G_{ik} = G_{\bar{i}\bar{k}} = 0,$$

Physically:

$$\int [dp][d\bar{p}][dX][d\bar{X}]e^{-\frac{1}{2\pi i h}\int_\Sigma(\langle p\wedge\bar{\partial}X\rangle - \langle\bar{p}\wedge\partial X\rangle - \langle g, p\wedge\bar{p}\rangle) + \int_\Sigma R^{(2)}(\gamma)\Phi_0(X)} =$$
$$\int [dX][d\bar{X}]e^{\frac{-1}{4\pi h}\int d^2z(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu + \int R^{(2)}(\gamma)(\Phi_0(X)+\log\sqrt{g})}$$

based on computations of
A. Tseytlin and A. Schwarz, Nucl.Phys. B399 (1993) 691-708.

# Main Conjecture

Consider

$$\mathbf{F}^{\cdot}_{b^-} = \mathcal{F}^{\cdot} \otimes \bar{\mathcal{F}}^{\cdot}|_{b^-=0}$$

with the $L_\infty$-algebra structure given by Lian-Zuckerman construction.

One can explicitly check that GMC symmetry
$(\Psi = \Psi(\mathbb{M}, \Phi, \text{auxiliary \ fields})$

$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \ldots,$$

reproduces

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}).$$

A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

Conjecture: The corresponding Maurer-Cartan equation gives Einstein equations on $G, B, \Phi$ expressed in terms of Beltrami-Courant differential. The symmetries of the Maurer-Cartan equation reproduce mentioned above symmetries of Einstein equations.

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Thank you!