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Field Equations from
Homotopy Algebras of CFT

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AMS Sectional Meeting (Raleigh, NC)

Special Session on Homotopical Algebra

with

Applications to Mathematical Physics

2009

- **Motivation**
- **Reminder of Lian-Zuckerman (LZ) homotopy algebras**
- **Relation of LZ constructions to perturbed CFTs and σ-models**
- **Examples:**
  - **Einstein equations**
  - **Yang-Mills equations**
  - **Kodaira-Spencer equations via chiral de Rham complex**
- **Conclusions**

Motivation

String theory:
2d Conformal Field Theory $\longrightarrow$
D-dimensional Quantum Field Theory

Linear classical field equations (Maxwell, Linearized Einstein) and their symmetries:

$$Q\Phi = 0, \qquad \Phi \longrightarrow \Phi + Q\lambda$$

Q **is a semi-infinite cohomology operator for the Virasoro algebra.**

What about nonlinear equations?

String Field Theory:

$$Q\Phi + \sum_n \mu_n\big(\Phi, \dots, \Phi\big) = 0$$
$$Q, \mu_n : A_\infty / L_\infty$$

The description of μ_n **is too complicated.**
Something more explicit?

Natural mathematical tool to study 2d CFT: VOA (vertex operator algebra)

B.H. Lian, G.J. Zuckerman
"New Perspectives on the BRST-algebraic Structure of String Theory"
Commun.Math.Phys.154 (1993) 613

LZ homotopy algebra (reminder)

Let V be the VOA. $T(z)$ **is a Virasoro element**
Semi-infinite complex : $C^* = V \otimes \Lambda$
Λ **- VOA of "conformal ghosts":**

$$b(z)c(w) \sim \frac{1}{z-w}$$

$$Q = \oint \mathrm{d}z \ c(z)T(z) + :c\partial cb:(z)$$

semi-infinite cohomology (BRST) operator.

Let $a(z)$ **be a vertex operator for** $a \in V \otimes \Lambda$

Operations:

$$\begin{aligned} \mu(a_1, a_2) &\equiv P_0 a_1(\varepsilon) a_2 \\ \{a_1, a_2\} &\equiv (-1)^{|a_1|} \oint \mathrm{d}z (b_{-1}a_1)(z) a_2 \end{aligned}$$

where P_0 **is the projection on** ε **- independent component.**

Proposition 1

$$\begin{array}{ll} (i) & \mu(a_1,a_2)-(-1)^{|a_1||a_2|}\mu(a_2,a_1)= \\ & Qm(a_1,a_2)+m(Qa_1,a_2)+(-1)^{|a_1|}m(a_1,Qa_2) \end{array}$$

$$\begin{array}{ll} (ii) & \mu\big(\mu(a_1,a_2),a_3\big)-\mu\big(a_1,\mu(a_2,a_3)\big)= \\ & Qn(a_1,a_2,a_3)+n(Qa_1,a_2,a_3)+(-1)^{|a_1|}n(a_1,Qa_2,a_3) \\ & +(-1)^{|a_1|+|a_2|}n(a_1,a_2,Qa_3), \end{array}$$

Lemma

$$(-1)^{|a_2|}\{a_1,a_2\}=b_0\mu(a_1,a_2)-\mu(b_0a_1,a_2)-(-1)^{|a_2|}\mu(a_1,b_0a_2)$$

Proposition 2

$$\begin{array}{ll} (i) & \{a_1,a_2\}+(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2,a_1\}= \\ & (-1)^{|a_1|-1}\big(Qm'(a_1,a_2)+m'(Qa_1,a_2) \\ & -(-1)^{|a_2|}m'(a_1,Qa_2)\big) \\ (ii) & \big\{\{a_1,a_2\},a_3\big\}-\big\{a_1,\{a_2,a_3\}\big\} \\ & +(-1)^{(|a_1|-1)(|a_2|-1)}\big\{a_2,\{a_1,a_3\}\big\}=0 \\ (iii) & \{a_1,\mu(a_2,a_3)\}=\mu(\{a_1,a_2\},a_3) \\ & +(-1)^{(|a_1|-1)||a_2|}\mu(a_2,\{a_1,a_3\}) \\ (iv) & \{\mu(a_1,a_2),a_3\}-\mu(a_1,\{a_2,a_3\}) \\ & -(-1)^{|(a_3|-1)|a_2|}\mu(\{a_1,a_3\},a_2)= \\ & (-1)^{|a_1|+|a_2|-1}\big(Qn''(a_1,a_2,a_3)-n''(Qa_1,a_2,a_3)- \\ & (-1)^{|a_2|}n''(a_1,Qa_2,a_3)-(-1)^{|a_1|+|a_2|}n''(a_1,a_2,Qa_3)\big) \end{array}$$

Therefore $V \otimes \Lambda$ carry a structure of homotopy Gerstenhaber algebra.
One can generalize further:

$$\begin{aligned} \mathbf{C}^* &= C^* \otimes \bar{C}^* \\ \mu^{\mathrm{ext}}(a_1, a_2) &= P_0 a_1(\epsilon) a_2, \qquad \epsilon \notin \mathbb{R} \\ \{a_1, a_2\}^{\mathrm{ext}} &= P_0 \oint_{C_{\varepsilon,0}} a_1^{(1)} a_2, \end{aligned}$$

where $a^{(1)} = \mathrm{d}z(b_{-1}a)(z) + \mathrm{d}\bar{z}(\bar{b}_{-1}a)(z)$**.**

$\mu^{\mathrm{ext}}, \{\cdot,\cdot\}^{\mathrm{ext}}$ **also satisfy the relations of homotopy Gerstenhaber algebra w.r.t. operator** $\mathbf{Q} = Q + \bar{Q}$

Conjecture (L-Z) Operations $\mu, \{\cdot,\cdot\}$ and $\mu^{\mathrm{ext}}, \{\cdot,\cdot\}^{\mathrm{ext}}$ can be extended to the structure of G_∞ algebra.

Attempts to prove:

Kimura, Voronov, Zuckerman
"Homotopy Gerstenhaber algebra and topological field theory"
q-alg/9602009
Galvez, Gorbounov, Tonks
"Homotopy Gerstenhaber Structures and Vertex Algebras"
math/0611231

What is the physical meaning of the associated Maurer-Cartan equations?

$$
\begin{aligned}
Q\Phi + \mu(\Phi,\Phi) + \sum_{n=3}^{\infty} \mu_n(\Phi,\ldots,\Phi) &= 0 \\
Q\Psi + \frac{1}{2}\{\Psi,\Psi\} + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Psi,\ldots,\Psi\}_n &= 0
\end{aligned}
$$

We will show that

a) **they lead to nonlinear field equations and their symmetries**

b) **can give rise to an "algebraic" definition of β-function for the perturbed CFT (σ-models in particular)**

Perturbed CFTs

VOA $\longrightarrow$ CFT with some action S_0.
Perturbations: $S_0 \longrightarrow S = S_0 + V$

$$V = \int_\Sigma \Phi^{(2)}, \quad \Phi^{(2)} = \mathrm{d}z \wedge \mathrm{d}\bar{z} A(z), \quad A \in C^* \otimes \bar{C}^*$$

In general, the perturbed theory is not a CFT. Renormalization theory gives the condition for the theory to be conformal:

$$\beta(\Phi) = 0$$

Expand: $\Phi^{(2)} = t\Phi_1^{(2)} + t^2\Phi_2^{(2)} + \dots$

$$\beta_1(V) = 0 \Longleftrightarrow Q\Phi_1^{(0)} = 0$$
$$\beta_2(V) = 0 \Longleftrightarrow Q\Phi_2^{(0)} + \frac{1}{4\pi i} P_0 \oint_{C_{\varepsilon,0}} \Phi_1^{(1)}\Phi_1^{(0)} = 0$$

where $\Phi_i^{(2)} = \mathrm{d}z \wedge \mathrm{d}\bar{z}[b_{-1}, [\bar{b}_{-1}, \Phi_i^{(0)}]]$

A. Sen (89,90);
A.M.Z.
"BRST, Generalized Maurer-Cartan Equations and CFT",
Nucl.Phys.B 794 (2006) 370,
"Formal Maurer-Cartan Structures: From CFT to Classical Field Equations",
JHEP0709:098(2007)

Therefore one can replace $\beta(\Phi)$ by

$$\hat{\beta}(\Phi) = Q\Phi + \frac{1}{2}\{\Phi, \Phi\} + \sum_{n=3}^{\infty} \frac{1}{n!}\{\Phi, \ldots, \Phi\}_n$$

where $\Phi^{(2)} = \mathrm{d}z \wedge \mathrm{d}\bar{z}[b_{-1}, [\bar{b}_{-1}, \Phi]]$

One can think of $\hat{\beta}(\Phi)$ as an "algebraic" definition of β-function.

The same applies to β-functions for the theories with boundary perturbations:

$$S_0 \longrightarrow S = S_0 + \int_{\partial\Sigma} \Phi^{(1)}$$

In this case "algebraic" β-function is:

$$\beta(\Phi) = Q\Phi + \mu(\Phi, \Phi) + \sum_{n=3}^{\infty} \mu_n(\Phi, \ldots, \Phi)$$

where $\Phi^{(1)} = \mathrm{d}z[b_{-1}, \Phi] + \mathrm{d}\bar{z}[\bar{b}_{-1}, \Phi]$

σ-models

$$S = \frac{1}{2\pi h}\int_\Sigma \mathrm{d}^2 z (G_{\mu\nu}(X) + B_{\mu\nu}(X))\partial X^\mu \bar{\partial} X^\nu$$
$$+\frac{1}{2\pi}\int R^{(2)}\phi(X) + \int_{\partial\Sigma} A_\mu(X)\mathrm{d}X^\mu$$
$$\beta(G, B, \phi, A) = \sum_{n=1}^{\infty} h^n \beta_n(G, B, \phi, A)$$

$\beta_1 = 0 \Longleftrightarrow$ **Einstein-Yang-Mills equations**
$\beta_n = 0$ $(n > 1)$ **gives equations with higher derivatives**

Complications:

i) $S = S_0 + V$ **destroys the geometric background:**

$$G_{\mu\nu} = \eta_{\mu\nu} + t g^{(1)}_{\mu\nu} + t^2 g^{(2)}_{\mu\nu} + \ldots$$

ii)

$$S_0 = \frac{1}{2\pi h}\int_\Sigma \mathrm{d}^2 z \eta_{\mu\nu}\partial X^\mu \partial X^\nu$$

$$X^\mu(z) X^\nu(w) \sim h\eta^{\mu\nu}\ln|z-w|^2$$

logarithmic VOA

Nevertheless, if we neglect logarithms Maurer-Cartan equation

$$Q\Phi + \frac{1}{2}\{\Phi, \Phi\} + \cdots = 0, \qquad \Phi(G, B, \Phi)$$

reproduces Einstein equations ($H = \mathrm{d}B$):

$$\begin{aligned}
&R_{\mu\nu} = -\frac{1}{4}H_{\mu\lambda\rho}H_{\nu}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi, \\
&\nabla_\mu H^{\mu\nu\rho} - 2(\nabla_\lambda\phi)H^{\lambda\nu\rho} = 0, \\
&4(\nabla_\mu\phi)^2 - 4\nabla_\mu\nabla^\mu\phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0
\end{aligned}$$

and their symmetries up to the second order in t.

A.M.Z.
"Formal Maurer-Cartan Structures: from CFT to Classical Field Equations"
JHEP0712:098(2007)

We need another formulation:

$$\begin{aligned}
S = \quad & \frac{1}{2\pi h}\int_\Sigma \mathrm{d}^2 z\big(p_i\bar\partial X^i + p_{\bar i}\partial X^{\bar i} - \\
& (g^{i\bar j}p_i p_{\bar j} + \mu^i_{\bar j}p_i\bar\partial X^{\bar j} + \\
& \mu^{\bar i}_j p_{\bar i}\partial X^j + b_{i\bar j}\partial X^i\bar\partial X^{\bar j})\big) + \int_\Sigma R^{(2)}\hat\phi
\end{aligned}$$

A.S. Losev, A.V. Marshakov, A.M.Z.
"On first order formalism in string theory"
Phys.Lett.B 633(2006)375

Complication: $g, b, \mu, \bar{\mu}, \hat{\phi} \longrightarrow G, B, \phi$ **nonlinear transformation:**

$$
\begin{aligned}
G_{s\bar{k}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_{s}\mu^{j}_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \\
B_{s\bar{k}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_{s}\mu^{j}_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}} \\
G_{si} &= -g_{i\bar{j}}\bar{\mu}^{\bar{j}}_{s} - g_{s\bar{j}}\bar{\mu}^{\bar{j}}_{i}, \\
G_{\bar{s}\bar{i}} &= -g_{\bar{s}j}\mu^{j}_{\bar{i}} - g_{\bar{i}j}\mu^{j}_{\bar{s}} \\
B_{si} &= g_{s\bar{j}}\bar{\mu}^{\bar{j}}_{i} - g_{i\bar{j}}\bar{\mu}^{\bar{j}}_{s}, \\
B_{\bar{s}\bar{i}} &= g_{\bar{i}j}\mu^{j}_{\bar{s}} - g_{\bar{s}j}\mu^{j}_{\bar{i}}, \\
\phi &= \log\sqrt{g} + \hat{\phi}.
\end{aligned}
$$

Advantage:

$$S_0 = \frac{1}{2\pi h}\int_{\Sigma} \mathrm{d}^2 z (p_i \bar{\partial} X^i + p_{\bar{i}} \partial X^{\bar{i}})$$

VOA:

$$X^i(z)p_j(w) \sim \frac{h\delta^i_j}{z-w}, \qquad X^{\bar{i}}(z)p_{\bar{j}}(w) \sim \frac{h\delta^{\bar{i}}_{\bar{j}}}{\bar{z}-\bar{w}}$$

When $\mu, \bar{\mu}, b = 0$

$$G_{i\bar{j}} = -B_{i\bar{j}} = g_{i\bar{j}}, \qquad \phi = \hat{\phi} + \log \sqrt{g}$$

Resulting Einstein equations appear to be bi-linear in $g^{i\bar{j}}$

$$\begin{aligned}
&\partial_i \partial_{\bar{k}} \Phi_0 = 0, \quad \partial_{\bar{p}} d_{\bar{l}}^{\Phi_0} g^{\bar{l}k} = 0, \quad \partial_p d_l^{\Phi_0} g^{\bar{k}l} = 0, \\
&2g^{r\bar{l}} \partial_r \partial_{\bar{l}} g^{i\bar{k}} - 2\partial_r g^{i\bar{p}} \partial_{\bar{p}} g^{r\bar{k}} - g^{i\bar{l}} \partial_{\bar{l}} d_s^{\Phi_0} g^{s\bar{k}} - \\
&g^{r\bar{k}} \partial_r d_{\bar{j}}^{\Phi_0} g^{\bar{j}i} + \partial_r g^{i\bar{k}} d_{\bar{j}}^{\Phi_0} g^{\bar{j}r} + \partial_{\bar{p}} g^{\bar{k}i} d_n^{\Phi_0} g^{n\bar{p}} = 0,
\end{aligned}$$

where $d_i^{\Phi_0} g^{i\bar{j}} \equiv \partial_i g^{i\bar{j}} - 2\partial_i \Phi_0 g^{i\bar{j}}$ **and** $d_{\bar{i}}^{\Phi_0} g^{\bar{i}j} \equiv \partial_{\bar{i}} g^{j\bar{i}} - 2\partial_{\bar{i}} \Phi_0 g^{j\bar{i}}$**.**

They are equivalent to:

$$Q\Phi + \frac{1}{2}\{\Phi, \Phi\} = 0, \quad [b_{-1}, [\bar{b}_{-1}, \Phi]] = g^{i\bar{j}} p_i p_{\bar{j}}$$

at the order h^1**.**
Symmetries (holomorphic):

$$\Phi \longrightarrow \Phi + Q\Lambda + \{\Phi, \Lambda\} + \{\Lambda, \Phi\}$$

A.M.Z.
"Perturbed β-γ Systems and Complex Geometry", Nucl.Phys.B 759(2006) 370

Yang-Mills equations

$$*d_A * F = 0, \qquad F = dA + A \wedge A$$

$$0 \longrightarrow \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}{*}\mathrm{d}} \Omega^{D-1} \xrightarrow{\mathrm{d}} \Omega^D \longrightarrow 0$$

Yang-Mills C_∞ algebra:

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{\mathcal{Q}_\eta} \mathcal{F}^1 \xrightarrow{\mathcal{Q}_\eta} \mathcal{F}^2 \xrightarrow{\mathcal{Q}_\eta} \mathcal{F}^3 \longrightarrow 0$$

$$\begin{aligned} (\cdot,\cdot) \ &: \ \mathcal{F}^i \otimes \mathcal{F}^j \longrightarrow \mathcal{F}^{i+j} \\ (\cdot,\cdot,\cdot) \ &: \ \mathcal{F}^i \otimes \mathcal{F}^j \otimes \mathcal{F}^k \longrightarrow \mathcal{F}^{i+j+k-1} \end{aligned}$$

$(f_1, f_2)=$

$f_2 \backslash f_1$	v	$\mathbf{A}$	$\mathbf{V}$	a
w	vw	$\mathbf{A}w$	$\mathbf{V}w$	aw
$\mathbf{B}$	$v\mathbf{B}$	$(\mathbf{A},\mathbf{B})$	$\mathbf{B}\wedge\mathbf{V}$	$\mathbf{0}$
$\mathbf{W}$	$v\mathbf{W}$	$\mathbf{A}\wedge\mathbf{W}$	$\mathbf{0}$	$\mathbf{0}$
b	vb	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

$$\begin{aligned} (\mathbf{A},\mathbf{B}) \ &= \ (\mathbf{A} \wedge *\mathrm{d}\mathbf{B}) - (\mathbf{B} \wedge *\mathrm{d}\mathbf{A}) + \mathrm{d} * (\mathbf{A} \wedge \mathbf{B}) \\ (\mathbf{A},\mathbf{B},\mathbf{C}) \ &= \ \mathbf{A} \wedge *(\mathbf{B} \wedge \mathbf{C}) - \mathbf{C} \wedge *(\mathbf{A} \wedge \mathbf{B}) \end{aligned}$$

$$*d_A * F = 0 \Longleftrightarrow \mathcal{Q}_\eta \mathbf{A} + (\mathbf{A},\mathbf{A}) + (\mathbf{A},\mathbf{A},\mathbf{A}) = 0$$

See e.g. A.M.Z.
"BV Yang-Mills as a Homotopy Chern-Simons via SFT", Int.J.Mod.Phys.A 24 (2009) 1309;
"Conformal Field Theory and Algebraic Structure of Gauge Theory", arXiv:0812.1840

This algebra can be obtained from

$$S = \frac{1}{2\pi h}\int_D \mathrm{d}^2 z \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu} + \int_{\partial D} A_\mu(X)\mathrm{d}X^\mu$$

Complex $(\mathcal{F}^*, \mathcal{Q}_\eta)$ is a subcomplex in the BRST complex (C^*, Q_η) of open string. LZ operations μ, ν reproduce operations $(\cdot,\cdot)$ and $(\cdot,\cdot,\cdot)$ at the level h^0.

In order to get rid of logarithms we need introduce first order formulation:

$$S = \frac{1}{2\pi h}\int_{H^+} \mathrm{d}^2 z (p_\mu \bar{\partial} X^\mu + \bar{p}_\mu \partial \bar{X}^\mu - \eta^{\mu\nu} p_\mu \bar{p}_\nu) +$$
$$\int_{\mathbb{R}} \mathrm{d}z (\mathcal{A}_\mu(X)\partial X^\mu + \mathcal{B}^\mu(X) p_\mu)$$

Boundary conditions:

$$p_\mu|_{\mathbb{R}} = \bar{p}_\mu|_{\mathbb{R}}, \;\; X^\mu|_{\mathbb{R}} = \bar{X}^\mu|_{\mathbb{R}}$$

VOA:

$$X^\mu(z) p_\nu(w) \sim \frac{h\delta^\mu_\nu}{z-w}$$

Gaussian integration:

$$F: \quad \mathcal{A}_\mu(X)\partial X^\mu + \mathcal{B}^\mu(X) p_\mu \to A_\mu(X)\mathrm{d}X^\mu,$$

where $A_\mu(X) = \mathcal{A}_\mu(X) + \eta_{\mu\nu}\mathcal{B}^\nu(X)$

Extended BRST operator of open string:

$$\hat{Q}_\eta = Q_{X,p} + \eta^{\alpha\beta}\mu(a_\alpha, \{a_\beta, \cdot\}), \quad F\hat{Q}_\eta = Q_\eta F$$

where $Q_{X,p}$ is a BRST operator for X-p VOA, and $a_\alpha = cp_\alpha$.

It is possible to deform μ w.r.t. $\eta^{\alpha\beta}$ in such a way that the new operation μ^η will be homotopy commutative and associative w.r.t. $\hat{\mathcal{Q}}_\eta$. The complex $(\mathcal{F}^*, \mathcal{Q}_\eta)$ lies in the kernel of L_0 (0th Virasoro mode of X-p VOA):

$$\begin{aligned}
&\rho_u = u(X), \quad \phi'_{\mathbf{A}} = cA_\mu(X)\partial X^\mu, \\
&\phi''_{\mathbf{B}} = c : B^\mu(X)p_\mu :, \\
&\phi_a = \partial c a(X), \quad \psi'_{\mathbf{V}} = c\partial c V_\mu(X)\partial X^\mu, \\
&\psi''_{\mathbf{W}} = c\partial c : W^\mu(X)p_\mu : \\
&\psi_b = c\partial^2 c b(X), \quad \chi_v = c\partial c\partial^2 c v(X).
\end{aligned}$$

Combining deformed μ^η, n^η we find that they reproduce YM C_∞ algebra on $(\mathcal{F}^*, \mathcal{Q}_\eta)$.

Relation to Courant/Dorfman algebroid:

$$\{\phi'_{\mathbf{A}} + \phi''_{\mathbf{B}}, \phi'_{\bar{\mathbf{A}}} + \phi''_{\bar{\mathbf{B}}}\} = h(\phi'_{L_B\bar{\mathbf{A}} - \mathrm{d}i_{\bar{\mathbf{B}}}\mathbf{A}} + \phi''_{[B,\bar{B}]_{Lie}})$$

A.M.Z.
"β-γ systems and the deformations of BRST operator", to appear

Chiral de Rham complex and Kodaira-Spencer Theory

VOA (V):

$$X^i(z)p_j(w) \sim \frac{1}{z-w}, \qquad \psi^i(z)\chi_j(w) \sim \frac{1}{z-w}$$

Chiral de Rham cohomology operator:
$Q = \frac{1}{2\pi i}\oint \psi^i p_i$

Bilinear operation $\{a,b\} = \frac{(-1)^{|a|}}{2\pi i}\oint \mathrm{d}z[Q,a](z)b$ **satisfies Gerstenhaber algebra together with** μ**.** $\{\cdot,\cdot\}$ **reproduces Schouten bracket on the operators** $f^{i_1\dots i_n}(x)\chi_{i_1}\dots\chi_{i_n}$

F. Malikov
"Lagrangian approach to Sheaves on Vertex Algebras"
Comm.Math.Phys. 278(2008) 487

Introducing antichiral part ($\bar{V}$):

$$X^{\bar{i}}(z)p_{\bar{j}}(w) \sim \frac{1}{\bar{z}-\bar{w}}, \qquad \psi^{\bar{i}}(\bar{z})\chi_{\bar{j}}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}$$

Let us consider elements of the form
$\xi^{i_1\dots i_n}_{\bar{j}_1\dots\bar{j}_n}(X,\bar{X})\chi_{i_1}\dots\chi_{i_n}\psi^{\bar{j}_1}\dots\psi^{\bar{j}_n} \in V\otimes\bar{V}$
Then
$\bar{Q}\mu+\{\mu,\mu\}=0$**,** $\mu = \mu^i_{\bar{j}}(X,\bar{X})\chi_i\psi^{\bar{j}}$ **coincides with Kodaira-Spencer equation.**

Actually, most of Barannikov-Kontsevich formulas

S. Barannikov, M. Kontsevich
"Frobenius Manifolds and Formality of Lie algebras of Polyvector Fields",
alg-geom/9710032

may be reproduced on VOA language.

Conclusions

- **Physical interpretation of "higher homotopies" in Lian-Zuckerman construction.**
- **Construction of the field theory equations (Einstein, YM, Kodaira-Spencer) via the deformation theory of semi-infinite cohomology operator for certain VOAs.**
- **Algebraic approach to the study of β-function in σ-models.**
- **Relation of Courant/Dorfman algebroid and Yang-Mills C_∞-algebra.**