# Homotopy relations for topological vertex algebras

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# Outline

Motivations

Short reminder of $A_\infty$-algebras

Construction of LZ $A_\infty$-algebra via polytopes

Final remarks/Open questions

# Motivations (physics)

Topological CFT and its perturbations

$Q^2 = 0 \qquad [Q, b] = T$ (energy-momentum tensor)

After perturbation: $Q \to Q^{\mathrm{def}}$

$$Q^{\mathrm{def}} V = QV + \sum_{n \geqslant 2} m_n(\Phi \dots \Phi, V)$$

Perturbation types:

1) $S + \int_\Sigma \Phi^{(2)} \qquad (\{m_n\}, Q)$ satisfy $L_\infty -$ algebra relations

2) $S + \int_{\partial\Sigma} \Phi^{(1)} \qquad (\{m_n\}, Q)$ satisfy $A_\infty -$ algebra relations

Condition ${Q^{\mathrm{def}}}^2 = 0 \Longleftrightarrow$
Generalized Maurer-Cartan equation

$$Q\Phi + m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \dots = 0$$

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In particular:
CFT and BRST (semi-infinite cohomology) complex
${Q^{\mathrm{def}}}^2 = 0$ for marginal deformations $\Longleftrightarrow \beta(\Phi) = 0$, i.e. conformal invariance

For $\sigma$-models of String Theory:

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# Motivations (Mathematics)



Homotopy algebras related to TVOA:

$$J(z)=\sum_n j_n z^{-n-1}, b(z)=\sum_n b_n z^{-n-2}, Q\equiv j_0:$$
$$[Q,b(z)]=L(z),\quad Q^2=0$$
$$N(z)=\sum N_n z^{-n-1}, N_0\quad \text{provides grading}$$

Lian-Zuckerman (1993):

$$(A,B)\equiv \mathrm{Res}_z\frac{A(z)B}{z},\qquad \{A,B\}\equiv \mathrm{Res}_z b_{-1}A(z)B$$

generate Gerstenhaber algebra on cohomology of $Q$.

Conjecture: (Q,(,),{,}) extends to $G_\infty$-algebra on the space of TVOA.

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The corresponding "$\beta$-functions" are the same modulo "field redefinitions"

$$\Phi \to \Phi + \alpha_2(\Phi, \Phi) + \alpha_3(\Phi, \Phi, \Phi) + \dots$$

$\{\alpha_n\}$ generate $A_\infty/L_\infty$ morpism.

Problem: How to find explicit expressions for higher operations of LZ homotopy algebra?

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# Short reminder of $A_\infty$-algebras





$A_\infty$-algebra $\mathbb{Z}$=graded space V with differential $Q$ and multilinear operations $\{m_i\}$, $(m_1 = Q)$ of degree $2-i$.

Relations:

$$Q^2 = 0$$
$$Qm_2(a,b) = m_2(Qa,b) + (-1)^{|a|} m_2(a,Qb)$$
$$m_2(m_2(a,b),c) - m_2(a,m_2(b,c)) = Qm_3(a,b,c) +$$
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In general, $\sum_{i=1}^{n-1}(-1)^i M_i \circ M_{n-i+1} = 0$ on $V^{\otimes^n}$ where $M_i = \sum_{k+\ell+i=n} \mathbf{1}^{\otimes^k} \otimes m_i \otimes \mathbf{1}^{\otimes^\ell}$

$A_\infty$-algebra is algebra over the operad, generated by Stasheff polytopes (associahedra) $K_n$:

$K_2 = \bullet$ $\quad$ $K_3 = \bullet$——$\bullet$ $\quad$ $K_4 =$ 

Each face of codimension one is $K_r \times K_s$, $n = r + s - 1$. The inclusion $\circ_i : K_r \times K_s \to K_{r+s-1}$ makes it a non-symmetric operad.

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# Construction of LZ $A_\infty$-algebra via polytopes

Let V be topological VOA.
We consider all vertex operators on $\mathbb{R}_+$.
Regularized LZ operation: $(A_1, A_2)_\varepsilon(t) = A_1(t+\varepsilon)A_2(t)$
Proposition Let $\varepsilon > 0, t > \varepsilon$.

$$A(t+\varepsilon)B(t) - (-1)^{|A||B|}B(t-\varepsilon)A(t) =$$
$$Qm_\varepsilon(A,B)(t) + m_\varepsilon(QA,B)(t) + (-1)^{|A|}m_\varepsilon(A,QB)(t),$$
$$m_\varepsilon(A,B)(t) = \int_{-\varepsilon}^{0}[b_{-1}, A(t'+t+\varepsilon)B(t'+t)]dt'$$

This is the regularized form of homotopy commutativity condition.

What about associativity?

$$A(\rho+t)B(\rho-\alpha_1+t)C(t) - A(\rho+t)B(\alpha_2+t)C(t) =$$
$$A(t+\rho)\int_{\alpha_2}^{\rho-\alpha_1}[L_{-1}, B(t'+t)]dt'\,C(t) =$$
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*Higher commutativity and associativity relations?*

*Framework:*
*Introduction of nonlocal operators*
*generated by 2 types of operations:*

$$I \quad : \quad A, B \to A(t+\varepsilon)B(t)$$
$$II \quad : \quad A_1,\dots,A_n \to \int_P A_1(t_1+t)\dots A_n(t_n+t)\mathrm{d}t_1\wedge\dots\wedge\mathrm{d}t_n$$
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$$(-1)^{|A|} n_{\rho,\alpha_1,\alpha_2}(A,QB,C)(t) + (-1)^{|A|+|B|} n_{\rho,\alpha_1,\alpha_2}(A,B,QC)(t),$$
$$n_{\rho,\alpha_1,\alpha_2}(A,B,C)(t) = (-1)^{|A|} \int_{\alpha_2}^{\rho-\alpha_1} A(t+\rho)[b_{-1},B](t'+t)C(t)dt'$$
$$+(m_{\alpha_1}(A,B),C)_\rho(t).$$

*Higher commutativity and associativity relations?*

*Framework:*
*Introduction of nonlocal operators*
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$$I \quad : \quad A,B \to A(t+\varepsilon)B(t)$$
$$II \quad : \quad A_1,\dots,A_n \to \int_P A_1(t_1+t)\dots A_n(t_n+t)\mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_n$$
$$\text{where} \qquad P \in \Delta_\rho = \{(t_1,\dots,t_n) \mid 0 < t_n < t_{n-1} \dots < t_1 < \rho\}$$

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# Higher commutativity relation



Proposition

$$n_{\rho,\varepsilon_1,\varepsilon_2}(A,B,C)(t)+(-1)^{|A||B|+|A||C|+|C||B|}n_{-\rho,-\varepsilon_2,-\varepsilon_1}(C,B,A)(t)=$$
$$(m_{\varepsilon_2}(A,B),C)_\rho(t)+(-1)^{|A||B|+|A||C|+|C||B|}(m_{-\varepsilon_2}(C,B),A)_{-\rho}(t)+$$
$$(-1)^{|A||B|}m_\rho((B,A)_{-\varepsilon_1},C)(t)-m_\rho(A,(B,C)_{\varepsilon_2})(t)+$$
$$Q\tilde{m}_{\rho,\varepsilon_1,\varepsilon_2}(A,B,C)(t)-\tilde{m}_{\rho,\varepsilon_1,\varepsilon_2}(QA,B,C)(t)$$

where $t>\rho>>\varepsilon_{1,2}$ and

$$\tilde{m}_{\rho,\varepsilon_1,\varepsilon_2}(A,B,C)(t)=$$
$$[b_{-1},\int_{-\rho}^{0}(-1)^{|A|}A(s+\rho+t)\int_{\varepsilon_2}^{\rho-\varepsilon_1}[b_{-1},B(t'+s+t)]dt'C(t+s)]ds].$$

# Pentagon relation

$$p'_P(A_1, A_2, A_3, A_4)(t) =$$
$$(-1)^{|A_2|} A_1(\rho + t) \int_P [b_{-1}, A_2](x + t)[b_{-1}, A_3(y + t)]dx \wedge dyA_4(t).$$

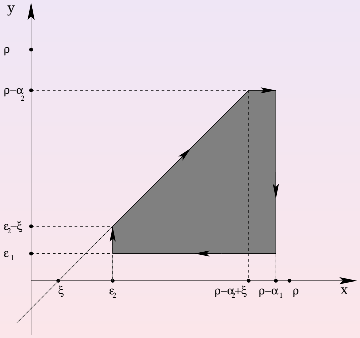


Figure: Pentagon $P$

$\rho >> \varepsilon_2, \alpha_2 > \varepsilon_1, \alpha_1, \xi > 0$

Higher homotopy:

$$p_P(A_1, A_2, A_3, A_4)(t) = p'_P(A_1, A_2, A_3, A_4)(t) +$$
$$n_{\rho,\alpha_2,\varepsilon_1}(m_{\alpha_1}(A_1, A_2), A_3, A_4)(t) + (\tilde{m}_{\alpha_2,\alpha_1,\xi}(A_1, A_2, A_3), A_4)_\rho$$

Proposition Operations $(\cdot,\cdot)$, $n(\cdot,\cdot,\cdot)$, $p(\cdot,\cdot,\cdot,\cdot)$ satisfy the following relation:

$$(-1)^{|A_1|}(A_1, n_{\varepsilon_2,\xi,\varepsilon_1}(A_2, A_3, A_4))_\rho - n_{\rho,\alpha_1,\varepsilon_2}(A_1, A_2, (A_3, A_4)_{\varepsilon_1}) +$$
$$n_{\rho,\alpha_2,\epsilon_2}(A_1, (A_2, A_3)_\xi, A_4)(t) - n_{\rho,\alpha_2,\epsilon_1}((A_1, A_2)_{\alpha_1}, A_3, A_4)(t) +$$
$$(n_{\alpha_2,\alpha_1,\xi}(A_1, A_2, A_3), A_4)_\rho(t) =$$
$$Qp_P(A_1, A_2, A_3, A_4)(t) - p_P(QA_1, A_2, A_3, A_4)(t) -$$
$$(-1)^{|A_1|}p_P(A_1, QA_2, A_3, A_4) - (-1)^{|A_1|+|A_2|}p_P(A_1, A_2, QA_3, A_4)(t) -$$
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This is a relation from $A_\infty$ algebra modulo the dependence on the parameters.

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We see the pattern:

$$m_\rho(A,B)(t) = \int_{-\rho}^{0} [b_{-1}, (A,B)_\rho(t+t')]dt',$$

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Conjecture: For $n \geq 3$

$$\mu_n^{\rho,K_n}(A_1,A_2,....,A_n)(t) = {\mu'}_n^{\rho,K_n}(A_1,A_2,....,A_n)(t),$$
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$K_n$ – (n-2)-dimensional Stasheff polytope
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Complete proof? One should use Shneider-Sternberg procedure in order to cut the associahedron from the simplex.
Relation to the compactification of the real moduli space $\bar{M}_{0,n}(\mathbb{R})$?
2) $L_\infty$ structure for "full" CFT?
Find explicit expressions for operations.
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