Supersymmetric extension of KdV equation, Drinfeld-Sokolov reduction and inverse scattering

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1 Introduction

$$
\begin{aligned}
\varepsilon_t &= -4\varepsilon_{xxx} + 6u\varepsilon_x + 3u_x\varepsilon \\
u_t &= -u_{xxx} + 6uu_x - 12\varepsilon\varepsilon_{xx}
\end{aligned} \tag{1}
$$

$$\varepsilon(x,t) \in \Lambda(n)_{\bar{1}}, \qquad u(x,t) \in \Lambda(n)_{\bar{0}}$$

For every $\eta \in \Lambda(n) = \Lambda(n)_{\bar{1}} + \Lambda(n)_{\bar{0}}$

$$\eta = \sum_{m\geq 0} \sum_{i_1<\ldots<i_m} \eta_{i_1\ldots i_m}\Theta^{i_1}\ldots\Theta^{i_m}$$

where $\quad \Theta_i\Theta_j + \Theta_j\Theta_i = 0 \quad \forall i,j.$

Equations (1) have Hamiltonian form

$$u_t = \{H, u\}$$

$$\varepsilon_t = \{H, \varepsilon\}$$

with the brackets:

$$
\begin{aligned}
\{\varepsilon(x), \varepsilon(y)\} &= -\frac{1}{2}(\delta''(x-y) - u(x)\delta(x-y)) \\
\{u(x), \varepsilon(y)\} &= -\frac{1}{2}(3\varepsilon(x)\delta'(x-y) + \varepsilon'(x)\delta(x-y)) \\
\{u(x), u(y)\} &= -\frac{1}{2}(\delta'''(x-y) - 2u'(x)\delta(x-y) \\
&\quad -4u(x)\delta'(x-y))
\end{aligned} \tag{2}
$$

and

$$\boldsymbol{H} = \int \boldsymbol{u}^2 - 4\varepsilon\varepsilon'$$

Also we can obtain such an equation

using L-A pair: $\boxed{\dot{\mathcal{L}} = [\boldsymbol{A}, \mathcal{L}]}$

$$\mathcal{L} = \partial_x + \begin{pmatrix} 0 & 0 & 1 \\ -\varepsilon & 0 & 0 \\ u+\lambda & \varepsilon & 0 \end{pmatrix} \tag{3}$$

$$\boldsymbol{A} = \tag{4}$$

$$\begin{pmatrix} -u_x & -4\varepsilon_x & 4\lambda - 2u \\ 2\varepsilon u - 4\varepsilon\lambda - 4\varepsilon_{xx} & 0 & -4\varepsilon_x \\ 4\varepsilon\varepsilon_x + u_{xx} - 2u^2 + 2u\lambda + 4\lambda^2 & -2\varepsilon u + 4\varepsilon\lambda + 4\varepsilon_{xx} & u_x \end{pmatrix}$$

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where this matrix operators belong to

Lie superalgebra osp(1/2):

$$\begin{array}{ll} [h, X^{\pm}] = \pm 2X^{\pm} & [h, v^{\pm}] = \pm v^{\pm} \\ [X^{+}, X^{-}] = h & [v^{+}, v^{-}] = -h \\ [X^{\pm}, v^{\pm}] = v^{\pm} & [v^{\pm}, v^{\pm}] = \pm 2X^{\pm} \end{array}$$

$v^{\pm}$ - odd; $\quad h, X^{\pm}$ - even.

2 Adler-Kostant-Symes construction and Drinfeld-Sokolov reduction for Osp(1/2)

AKS approach associates

to the L-operator

$$\boxed{\mathcal{L} = \partial_x + J(x) + \lambda X_- \qquad (5)}$$

$(J(x) \in osp(1/2))$ such Poisson brackets:

$$\begin{aligned} \{\phi, \psi\}_1(J) &= -str \int X_-[\frac{\delta\phi}{\delta J(z)}, \frac{\delta\psi}{\delta J(z)}] \\ \{\phi, \psi\}_2(J) &= str \int J(z)[\frac{\delta\phi}{\delta J(z)}, \frac{\delta\psi}{\delta J(z)}] \\ &\quad + str \int \partial_z(\frac{\delta\phi}{\delta J(z)})\frac{\delta\psi}{\delta J(z)}) \end{aligned} \qquad (6)$$

The relation:

The gauge transformation of L-operator $\mathcal{L} = \partial_x + \mu$:

$$\mathcal{L} = \partial_x + \mu \rightarrow g\mathcal{L}g^{-1} = \partial_x + g\mu g^{-1} - g'g^{-1},$$

can be treated as coadjoint representation of central extended osp(1/2) superalgebra:

$$osp(1/2)_c = \{(X, a)|X : S^1 \rightarrow osp(1/2); a \in \mathrm{C}\}$$

$$[(X, a), (Y, b)] = (XY - YX, str \int_{S^1} dx X'(x)Y(x))$$

and L-operator itself can be treated as an element of the

$osp(1/2)^*_c$: $\mathcal{L} = (\mu, 1)$, and the pairing is defined in such a way:

$$< (\mu, e), (X, a) >= ea + str \int \mu X$$

The affinization via spectral parameter of g_c:

$$\begin{aligned} \ell(osp(1/2)_c) &= osp(1/2)_c \otimes C[\lambda, \lambda^{-1}] \\ &= \{\sum_{i=r}^{s} X_i \lambda^i | X_i \in osp(1/2)_c\} \end{aligned}$$

Pairing in this space has the form:

$$< \alpha, x >=< \sum_i \alpha_i \lambda^i, \sum_j x_j \lambda^j >= \sum_{i+j=-1} \alpha_i x_j = (\alpha(x))_{-1},$$
$$\alpha_i \in osp(1/2)^*_c, \; x_i \in osp(1/2)_c$$

Then we consider the Poisson subspace in $\ell(osp(1/2)^*_c)$ which is generated by operators like (5). and R-matrix Lie-Poisson bracket on it:

$$\{\phi, \psi\}_{R_\eta}(\alpha) =< \alpha, [\nabla_\alpha \phi, \nabla_\alpha \psi]_{R_\eta} > \tag{7}$$
$$\frac{1}{2}[R_\eta X, Y] + \frac{1}{2}[X, R_\eta Y] = [X, Y]_{R_\eta},$$

where $R = P_+ - P_-$, $R_\eta = R \circ \hat{\eta}$, $\eta = e_0 + e_1 \lambda$. This bracket will coincide with

$$\{\phi, \psi\}_{R_\eta}(J) = -e_0 str \int C_1 [\frac{\delta \phi}{\delta J(z)}, \frac{\delta \psi}{\delta J(z)}]$$
$$+e_1 (str \int J(z) [\frac{\delta \phi}{\delta J(z)}, \frac{\delta \psi}{\delta J(z)}] + str \int \partial_z (\frac{\delta \phi}{\delta J(z)}) \frac{\delta \psi}{\delta J(z)}))$$

This is the linear combination of the brackets (6). This brackets are invariant under gauge transformations generated by v_- and X_-

Next we consider operator:

$$\boxed{\begin{array}{l} \mathcal{L} = \partial_x + J(x) + \lambda X_- \\ str((J(x) - X^+)X_-) = 0 \end{array}} \qquad (8)$$

This means that in the matrix language it has the form:

$$\mathcal{L} = \partial_x + \begin{pmatrix} q(x) & \beta(x) & 1 \\ -\alpha(x) & 0 & \beta(x) \\ \lambda + w(x) & \alpha(x) & -q(x) \end{pmatrix} \qquad (9)$$

It makes sense the linear problem:

$$\mathcal{L}\Psi = 0$$

The upper term in the column is invariant under gauge transformation generated by v_- and X_-:

$$exp(\chi v_- \ + \ pX_-)\Psi = \qquad (10)$$
$$\begin{pmatrix} 1 & & \\ -\chi & 1 & \\ p & \chi & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = \tilde{\Psi}$$

Using gauge transformation we can reduce(in a hamiltonian way) our “general” operator (5) to the lower triangular form (3). Matrix problem (3) is equal to scalar linear problem:

$$\boxed{(-\partial_x^2 + \lambda + u(x) - \varepsilon(x)\partial^{-1}\varepsilon(x)))\psi_1(x, \lambda) = 0} \qquad (11)$$

In the case of one-dimensional Grassmann algebra this is the Sturm-Liouville equation. Poisson brackets on u(x) and $\varepsilon(x)$ are:

$$
\begin{aligned}
\{\varepsilon(x),\varepsilon(y)\}_2 &= -\frac{1}{2}(\delta''(x-y)-u(x)\delta(x-y)) \\
\{u(x),\varepsilon(y)\}_2 &= -\frac{1}{2}(3\varepsilon\delta'(x-y)+\varepsilon'\delta(x-y)) \qquad (12)\\
\{u(x),u(y)\}_2 &= -\frac{1}{2}(\delta'''(x-y)-2u'(x)\delta(x-y) \\
&\quad -4u(x)\delta'(x-y))
\end{aligned}
$$

$$
\begin{aligned}
\{\varepsilon(x),\varepsilon(y)\}_1 &= -\frac{1}{2}\delta(x-y) \\
\{u(x),\varepsilon(y)\}_1 &= 0 \qquad (13)\\
\{u(x),u(y)\}_1 &= -2\delta'(x-y)
\end{aligned}
$$

Expanding $\Psi(x,\lambda)$ on λ we get infinite sequence of commuting Hamiltonians. Here are a few of them:

$$
\begin{aligned}
H_1 &= \int u(x)dx \\
H_2 &= \int(u^2(x)-4\varepsilon(x)\varepsilon'(x))dx \\
H_3 &= \int((u')^2+2u^3-16\varepsilon'(x)\epsilon''(x)-24\varepsilon(x)\varepsilon'(x)u(x)dx
\end{aligned}
$$

There is an interesting property:

$$\{H_{i+1},\phi\}_1=\{H_i,\phi\}_2$$

3 Inverse Scattering Transform for sKdV equation

$$\boxed{\begin{aligned} \mathcal{L}\psi &= 0 \\ \partial_t\psi &= A\psi \end{aligned}}$$

L-operator can be transformed in such a way

$$\tilde{\mathcal{L}}(x) = \partial_x + \begin{pmatrix} -ik & & \\ & 0 & \\ & & ik \end{pmatrix} + \begin{pmatrix} -u(2ik)^{-1} & -\varepsilon(2ik)^{-1} & -u(2ik)^{-1} \\ -\varepsilon & 0 & -\varepsilon \\ u(2ik)^{-1} & \varepsilon(2ik)^{-1} & u(2ik)^{-1} \end{pmatrix}$$

In this diagonalized form the first of equations can be rewritten this way:

$$\boxed{\partial_x\Psi = ikh\Psi + Q(x,k)\Psi \qquad (14)}$$

where

$$Q(x,k) = \begin{pmatrix} u(2ik)^{-1} & \varepsilon(2ik)^{-1} & u(2ik)^{-1} \\ \varepsilon & 0 & \varepsilon \\ -u(2ik)^{-1} & -\varepsilon(2ik)^{-1} & -u(2ik)^{-1} \end{pmatrix} \qquad (15)$$

Then we should consider matrix Jost solutions:

$$\Phi^+(x,k) = \begin{pmatrix} e^{ikx} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-ikx} \end{pmatrix} \qquad x \to \infty, \qquad (16)$$

$$\Phi^-(x,k) = \begin{pmatrix} e^{ikx} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-ikx} \end{pmatrix} \qquad x \to -\infty$$

and so-called transfer matrix: $\Phi^+(x,k)T(k) = \Phi^-(x,k)$

$$\boxed{\boldsymbol{T(k)} = \begin{pmatrix} a & \gamma & b \\ \bar{\delta} & f & \delta \\ \bar{b} & \bar{\gamma} & \bar{a} \end{pmatrix} \qquad (17)}$$

(bar means inverting sign of k and complex conjugation if k lies on real axis)

The conditions on elements of T(k) are

$$\begin{aligned} f &= 1 + 2ik\bar{\gamma}\gamma \\ f(a\bar{a} - b\bar{b}) &= 1 \\ \delta &= 2ik(\bar{\gamma}b - \gamma\bar{a}) \end{aligned}$$

And in the case of 1-dimensional Grassmann algebra:

$$f = 1, \quad a\bar{a} - b\bar{b} = 1, \quad \delta = 2ik(\bar{\gamma}b - \gamma\bar{a})$$

The integral equation for the first column of Jost matrix $\Phi^{+}(x,k)$, multiplied by e^{-ikx} and transformed back to the $\mathcal{L}$ problem is:

$$\boxed{\begin{array}{l} \tilde{m}_{+}(x,k) = \qquad\qquad\qquad\qquad\qquad\qquad (18) \\ \begin{pmatrix} 1 + \int_x^\infty dy \frac{e^{-2ik(x-y)}-1}{2ik}(\tilde{m}_{+}^{1}(y,k)u(y) - \varepsilon(y)\tilde{m}_{+}^{2}(y,k)) \\ - \int_x^\infty dy e^{-ik(x-y)}\varepsilon(y)\tilde{m}_{+}^{1}(y,k) \\ \int_x^\infty dy e^{-2ik(x-y)}(u(y)\tilde{m}_{+}^{1}(y,k) - \varepsilon(y)\tilde{m}_{+}^{2}(y,k)) \end{pmatrix} \end{array}}$$

We can see that it is analitic in upper half plane.

The same way we can consider $\tilde{m}_{-}(x,k)$,obtained from the first column of $\Phi^{-}(x,k)$,it will be analitic in the lower half plane.

From the transfer matrix relation between the columns of Jost matrices we can obtain an identity:

$$\boxed{m_+(x,k) - m_-(x,k) = V(k,x)m_-(x,-k) + f(x,k)} \quad (19)$$

where

$$V(k,x) = -r(k)e^{-2ikx}P(k),$$
$$f(x,k) = -2ikN^{-1}(k)\rho(k)\Phi^{-2}(x,k)e^{-ikx}$$
$$r(k) = \frac{\bar{b}(k)}{\bar{a}(k)}, \qquad \rho(k) = \frac{\bar{\gamma}(k)}{\bar{a}(k)}$$
$$m_+(x,k) = \frac{\tilde{m}_+(x,k)}{\bar{a}(k)}, \quad m_-(x,k) = \tilde{m}_-(x,k)$$
$$u(x) = 2ik\partial_x \tilde{m}_+^1(x,k), \qquad \varepsilon(x) = ik\tilde{m}_+^2(x,k), \qquad |k| \to \infty \quad (20)$$

In one dimensional case of Grassmann algebra f(x,k)=f(k), it is a constant.

The scheme for IST is:

$$\boxed{\varepsilon(x,0), u(x,0) \xrightarrow{\text{I}} s(0) \xrightarrow{\text{II}} s(t) \xrightarrow{\text{III}} u(x,t), \varepsilon(x,t)}$$

where s is the scattering data:

$$\{i\kappa_\ell\}, \{b(i\kappa_\ell)\}, \{\gamma(i\kappa_\ell)\}, \rho(k), r(k)$$

Evolution :

$$\begin{aligned} \dot{\bar{a}}(k,t) &= 0 \\ \dot{\bar{\gamma}}(k,t) &= -4ik^3\bar{\gamma}(k,t) \\ \dot{\bar{b}}(k,t) &= -8ik^3\bar{b}(k,t) \end{aligned}$$

Integral equations for Inverse Problem are:

$$
\begin{aligned}
R_n(x) &= -\frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)}(1 + i\sum_{j=1}^{N}\frac{R_j(x)}{\kappa_n+\kappa_j} \\
&\quad -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dz \frac{r(z)R(-z,x)e^{-2izx}}{z-i\kappa_n}) \\
R(x) &= 1+\sum_{j=1}^{N}\frac{R_j(x)}{k-i\kappa_j} - \int_{-\infty}^{+\infty} dz \frac{r(z)R(-z,x)e^{-2izx}}{z-(k+i0)} \\
\Theta_n(x) &= \frac{2\kappa_n\bar{\gamma}_n e^{\kappa_n x}}{\bar{a}'(i\kappa_n)} - -\frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)}(i\sum_{j=1}^{N}\frac{\Theta_j(x)}{\kappa_n+\kappa_j} \\
&- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dz \frac{r(z)\Theta(-z,x)e^{-2izx}+2iz\rho(z)e^{-izx}}{z-i\kappa_n}) \\
\Theta(x) &= \sum_{j=1}^{N}\frac{\Theta_j(x)}{k-i\kappa_j} - \\
&\quad \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dz \frac{r(z)\Theta(-z,x)e^{-2izx}+2iz\rho(z)e^{-izx}}{z-(k+i0)}
\end{aligned} \tag{21}
$$

where R(x) and $\Theta(x)$ are upper and middle elements of $m_+(x,k)$ correspondingly. Using them we can derive equations for the potentials:

$$
\begin{aligned}
u(x) &= \partial_x(2i\sum_{j=1}^{N}R_j(x) \\
&\quad +\frac{1}{\pi}\int_{-\infty}^{+\infty} dz(r(z)R(-z,x)e^{-2izx})) \\
\varepsilon(x) &= i\sum_{j=1}^{N}\Theta_j(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz(r(z)\Theta(-z,x)e^{-2izx} \\
&\quad +2iz\rho(z)e^{-izx})
\end{aligned} \tag{22}
$$

To get solitonic solutions we take r(k)and $\rho(k)$ equal to zero, so integral equations reduce to algebraic.

$$\boxed{\begin{aligned} R_n(x) &= -\frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)}\left(1 + i \sum_{j=1}^{N} \frac{R_j(x)}{\kappa_n + \kappa_j}\right) \\ \Theta_n(x) &= \frac{2\kappa_n \bar{\gamma}_n e^{\kappa_n x}}{\bar{a}'(i\kappa_n)} - i\frac{\bar{b}_n e^{2\kappa_n x}}{\bar{a}'(i\kappa_n)} \sum_{j=1}^{N} \frac{\Theta_j(x)}{\kappa_n + \kappa_j} \\ u(x) &= 2i\partial_x \sum_{j=1}^{N} R_j(x) \\ \varepsilon(x) &= i \sum_{j=1}^{N} \Theta_j(x) \end{aligned}}$$

As an example we can obtain 1-solitonic solutions:

$$u(x) = -\frac{c}{2ch^2(\frac{\sqrt{c}}{2}(x-ct))}, \qquad \varepsilon(x) = -\frac{\alpha}{ch(\frac{\sqrt{c}}{2}(x-ct))} \tag{23}$$

where α is any odd number.

Conclusions

1. **Considered AKS scheme and Drinfeld-Sokolov reduction for osp(1/2), as a result we obtained Poisson brackets and sKdV equation.**
2. **It is considered direct problem for associated linear problem for sKdV.**
3. **In the case of 1-dimensional Grassmann algebra it is considered inverse problem and N-solitonic solutions are obtained.**