Superconformal field theory and quantum inverse scattering method

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Abstract

The integrable structure of the two dimensional superconformal field theory is considered. The classical counterpart of our constructions is based on the $\widehat{osp}(1|2)$ super-KdV hierarchy. The quantum version of the monodromy matrix associated with the linear problem for the corresponding L-operator is introduced. Using the explicit form of the irreducible representations of $\widehat{osp}_q(1|2)$, the so-called "fusion relations" for the transfer matrices (that is, the traces of the corresponding monodromy matrices) considered in different representations of $\widehat{osp}_q(1|2)$ are obtained. The possible integrable perturbations of the model (primary operators, commuting with integrals of motion) are classified and the relation with the supersymmetric $\widehat{osp}(1|2)$ Toda field theory is discussed.

1 Introduction

Conformal field theory (CFT) provides effective tools to classify the fields in the theory we study and to compute their correlation functions. Perturbartion leads the system out of the critical point and breaks the conformal invariance. But special perturbations, called "integrable" still preserve an infinite-dimensional abelian algebra of conserved charges, thus leading to an integrable theory.

The authors of [1], [2] showed that in this case the problem could be studied from a point of view of continuous field theory version of the quantum inverse scattering method (QISM) [3],[4]. The proposition is the following: at first, to use CFT symmetries for construction of QISM structures at the scale invariant fixed point and then to study the integrable perturbed model by obtained QISM tools. Our object of study is a model based on the superconformal symmetry (see e.g. [5],[6]). The superconformal field theory has been applied to the physics of 2D disordered systems, to the study of lattice models (the tricritical Ising model) and to the superstring physics. We build basic structures of QISM: quantum monodromy matrices, RTT-relation and fusion relations for the transfer matrices, using underlying superconformal symmetry.

The $\widehat{osp}(1|2)$ supersymmetric Korteweg - de Vries theory (super-KdV) [8]-[11] is used as a classical limit of our quantum system. Due to the Drinfeld-Sokolov reduction of the $\widehat{osp}(1|2)$ affine superalgebra, Miura transformation and Poisson brackets are introduced. Then the monodromy matrix is constructed. The associated auxiliary L-matrix satisfies the r-matrix quadratic Poisson bracket relation and plays a crucial role in the following.

After this necessary preparation, we move to the quantum theory (Sec. 3). The quantum Miura transformation is realized by the free field representation of the superconformal algebra [5],[6]. Then the super-Virasoro module is considered, where the quantum versions of integrals of motion (IM) act. Vertex operators are also introduced, to build quantum monodromy matrix.

The algebraic structure corresponding to the quantum case, coincides with the quantum superalgebra $\widehat{osp}_q(1|2)$. We construct the finite dimensional irreducible representations of this quantum Lie superalgebra (Sec. 4). It appears, that in quantum monodromy matrix (in comparison with the classical case) one term is missing in the P-exponent. The quantum L-matrix satisfies the so-called RTT-relation, giving the integrability condition in the quantum case. Considering monodromy matrices in the obtained $\widehat{osp}_q(1|2)$ representations, we get the functional relations ("fusion relations") for their traces – "transfer matrices". When the deformation parameter is rational (the case of CFT minimal models), these fusion relations become the closed system of equations, which, due to the conjecture of [1] can be used to find the full set of eigenvalues of transfer matrices. Also, we suppose that they could be transformed to the Thermodynamic Bethe Ansatz equations [7].

Finally, we discuss integrable perturbations of the model and the relation with the supersymmetric Toda field theory.

2 A review of classical super-KdV theory

The classical limit of constructions of papers [1], [2] leads to the Drinfeld-Sokolov KdV hierarchies related to the corresponding affine Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$. Our quantum model gives in classical limit the super-KdV hierarchy [8]-[11] related to $B(0,1)^{(1)}$ (or $\widehat{osp}(1|2)$) affine Lie superalgebra. The supermatrix L-operator, corresponding to super-KdV theory is the following one:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Psi h - (iv_+\sqrt{\lambda} - \theta\lambda X_-), \tag{1}$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative, variable u lies on a cylinder of circumference 2π, θ is a Grassmann variable, $\Psi(u,\theta) = \phi(u) - i\theta\xi(u)/\sqrt{2}$ is a bosonic superfield; h, v_+, v_-, X_-, X_+ are generators of $osp(1|2)$ (for more information see [12], [13]):

$$[h, X_\pm] = \pm 2X_\pm, \quad [h, v_\pm] = \pm v_\pm, \quad [X_+, X_-] = h, \tag{2}$$

$$[v_\pm, v_\pm] = \pm 2X_\pm, \quad [v_+, v_-] = -h, \quad [X_\pm, v_\mp] = v_\pm, \quad [X_\pm, v_\pm] = 0.$$

Here [,] means supercommutator: $[a,b] = ab - (-1)^{p(a)p(b)} ba$ and the parity p is defined as follows: $p(v_\pm) = 1$, $p(X_\pm) = 0$, $p(h) = 0$. The "fermionic" operator $\mathcal{L}_F$ considered together with a linear problem $\mathcal{L}_F \chi(u,\theta) = 0$ is equivalent to the "bosonic" one:

$$\mathcal{L}_B = \partial_u - \phi'(u)h - \sqrt{\lambda/2}\,\xi(u)v_+ - \lambda(X_+ + X_-). \tag{3}$$

The fields ϕ, ξ satisfy the following boundary conditions:

$$\phi(u+2\pi) = \phi(u) + 2\pi i p, \quad \xi(u+2\pi) = \pm\xi(u), \tag{4}$$

where "+" corresponds to the so-called Ramond (R) sector of the model and "–" to the Neveu-Schwarz (NS) one. The Poisson brackets, given by the Drinfeld-Sokolov construction are the following:

$$\{\xi(u), \xi(v)\} = -2\delta(u-v), \quad \{\phi(u), \phi(v)\} = \frac{1}{2}\epsilon(u-v). \tag{5}$$

The L-operators (1), (3) correspond to the super-modified KdV, they are written in the Miura form. Making a gauge transformation to proceed to the super-KdV L-operator one obtains two fields:

$$U(u) = -\phi''(u) - \phi'^2(u) - \frac{1}{2}\xi(u)\xi'(u), \quad \alpha(u) = \xi'(u) + \xi(u)\phi'(u), \tag{6}$$

which generate the superconformal algebra under the Poisson brackets:

$$\begin{aligned}
\{U(u), U(v)\} &= \delta'''(u-v) + 2U'(u)\delta(u-v) + 4U(u)\delta'(u-v), \\
\{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u-v) + \alpha'(u)\delta(u-v), \\
\{\alpha(u), \alpha(v)\} &= 2\delta''(u-v) + 2U(u)\delta(u-v).
\end{aligned} \tag{7}$$

These brackets describe the second Hamiltonian structure of the super-KdV hierarchy. One can obtain an evolution equation by taking one of the corresponding infinite set of local IM (they could be obtained by expanding $\log(\mathbf{t}_1(\lambda))$, where $\mathbf{t}_1(\lambda)$ is the supertrace of the monodromy matrix, see below):

$$\begin{aligned}
I_1^{(cl)} &= \int U(u)\mathrm{d}u, \quad I_3^{(cl)} = \int \Big(\frac{U^2(u)}{2} + \alpha(u)\alpha'(u)\Big)\mathrm{d}u, \\
I_5^{(cl)} &= \int \Big((U')^2(u) - 2U^3(u) + 8\alpha'(u)\alpha''(u) + 12\alpha'(u)\alpha(u)U(u)\Big)\mathrm{d}u, \\
&\cdot \quad \cdot \quad \cdot
\end{aligned} \tag{8}$$

These conservation laws form an involutive set under the Poisson brackets: $\{I_{2k-1}^{(cl)}, I_{2l-1}^{(cl)}\} = 0$. From the $I_3^{(cl)}$ conservation law one obtains the super-KdV equation for the Grassmann algebra valued functions [8]-[10]:

$$U_t = -U_{uuu} - 6UU_u - 6\alpha\alpha_{uu}, \quad \alpha_t = -4\alpha_{uuu} - 6U\alpha_u - 3U_u\alpha. \tag{9}$$

Now let's consider the "bosonic" linear problem $\pi_s(\mathcal{L}_B)\chi(u) = 0$, where π_s means irreducible representation of $osp(1|2)$ labeled by an integer $s \geq 0$ [12],[13]. We can write the solution of this problem in such a way:

$$\chi(u) = \pi_s(\lambda)\Bigg(e^{-\phi(u)h_{\alpha_0}} P\exp\int_0^u \mathrm{d}u'\Big(\xi(u')e^{-\phi(u')}e_\alpha + e^{-2\phi(u')}2e_\alpha^2 + e^{2\phi(u')}e_{\alpha_0}\Big)\Bigg)\chi_0, \tag{10}$$

where $P\exp$ means P-ordered exponent, $\chi_0 \in C^{2s+1}$ is a constant vector and e_α, e_{α_0}, h_{α_0} are the Chevalley generators of $\widehat{osp}(1|2)$ (see [14]), which coincide in the evaluation representations $\pi_s(\lambda)$ with $\sqrt{\lambda/2}v_+$, λX_-, $-h$ correspondingly. The associated monodromy matrix then has the form:

$$\mathbf{M}_s(\lambda) = \pi_s\Bigg(e^{-2\pi i p h_{\alpha_0}} P\exp\int_0^{2\pi} \mathrm{d}u\Big(\xi(u)e^{-\phi(u)}e_\alpha + e^{-2\phi(u)}2e_\alpha^2 + e^{2\phi(u)}e_{\alpha_0}\Big)\Bigg). \tag{11}$$

Following [1] let's introduce auxiliary matrices: $\pi_s(\lambda)(\mathbf{L}) = \mathbf{L}_s(\lambda) = \pi_s(\lambda)(e^{\pi i p h_{\alpha_0}})\mathbf{M}_s(\lambda)$. They satisfy Poisson bracket algebra [15]:

$$\{\mathbf{L}_s(\lambda) \otimes, \mathbf{L}_{s'}(\mu)\} = [\mathbf{r}_{ss'}(\lambda\mu^{-1}), \mathbf{L}_s(\lambda) \otimes \mathbf{L}_{s'}(\mu)], \tag{12}$$

where $\mathbf{r}_{ss'}(\lambda\mu^{-1}) = \pi_s(\lambda) \otimes \pi_{s'}(\mu)(\mathbf{r})$ is the classical trigonometric $\widehat{osp}(1|2)$ r-matrix [16] taken in the corresponding representations:

$$\begin{aligned}\mathbf{r}(\lambda\mu^{-1}) = &\frac{1}{2}\frac{\lambda\mu^{-1} + \lambda^{-1}\mu}{\lambda\mu^{-1} - \lambda^{-1}\mu} h \otimes h + \\ &+ \frac{2}{\lambda\mu^{-1} - \lambda^{-1}\mu}(X_+ \otimes X_- + X_- \otimes X_+) \\ &+ \frac{1}{(\lambda\mu^{-1} - \lambda^{-1}\mu)}\left(\sqrt{\frac{\mu}{\lambda}}v_+ \otimes v_- - \sqrt{\frac{\lambda}{\mu}}v_- \otimes v_+\right).\end{aligned} \tag{13}$$

From the Poisson brackets for $\mathbf{L}_s(\lambda)$ one obtains that the traces of monodromy matrices $\mathbf{t}_s(\lambda) = str\mathbf{M}_s(\lambda)$ commute under the Poisson bracket: $\{\mathbf{t}_s(\lambda), \mathbf{t}_{s'}(\mu)\} = 0$. If one expands $\log(\mathbf{t}_1(\lambda))$ in the λ^{-1} power series, one can see that the coefficients in this expansion are the local IM, as we mentioned earlier.

3 Free field representation of Superconformal algebra and Vertex operators

To quantize the introduced classical quantities, we start from a quantum version of the Miura transformation (6), the so-called free field representation of the superconformal algebra [5]:

$$
\begin{aligned}
-\beta^2 T(u) &= :\phi'^2(u): + (1-\beta^2/2)\phi''(u) + \frac{1}{2} :\xi\xi'(u): + \frac{\epsilon\beta^2}{16} \\
\frac{i^{1/2}\beta^2}{\sqrt{2}} G(u) &= \phi'\xi(u) + (1-\beta^2/2)\xi'(u),
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
&\phi(u) = iQ + iPu + \sum_n \frac{a_{-n}}{n} e^{inu}, \qquad \xi(u) = i^{-1/2} \sum_n \xi_n e^{-inu}, \\
&[Q,P] = \frac{i}{2}\beta^2, \quad [a_n, a_m] = \frac{\beta^2}{2} n \delta_{n+m,0}, \qquad \{\xi_n, \xi_m\} = \beta^2 \delta_{n+m,0}.
\end{aligned}
\tag{15}
$$

Recall that there are two types of boundary conditions on ξ: $\xi(u+2\pi) = \pm\xi(u)$. The sign "+" corresponds to the R sector,the case when ξ is integer modded, the "–" sign corresponds to the NS sector and ξ is half-integer modded. The variable ϵ in (13) is equal to zero in the R case and equal to 1 in the NS case. One can expand $T(u)$ and $G(u)$ by modes in such a way: $T(u) = \sum_n L_{-n} e^{inu} - \frac{\hat{c}}{16}$, $G(u) = \sum_n G_{-n} e^{inu}$, where $\hat{c} = 5 - 2(\frac{\beta^2}{2} + \frac{2}{\beta^2})$ and L_n, G_m generate the superconformal algebra:

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n,-m} \\
[L_n, G_m] &= (\frac{n}{2} - m)G_{m+n} \\
[G_n, G_m] &= 2L_{n+m} + \delta_{n,-m}\frac{\hat{c}}{2}(n^2 - 1/4).
\end{aligned}
\tag{16}
$$

In the classical limit $c \to -\infty$ (the same is $\beta^2 \to 0$) the following substitution: $T(u) \to -\frac{\hat{c}}{4}U(u)$, $G(u) \to -\frac{\hat{c}}{2\sqrt{2i}}\alpha(u)$, $[,] \to \frac{4\pi}{i\hat{c}}\{,\}$ reduce the above algebra to the Poisson bracket algebra of super-KdV theory.

Let now F_p be the highest weight module over the oscillator algebra of a_n, ξ_m with the highest weight vector (ground state) $|p\rangle$ determined by the eigenvalue of P and nilpotency condition of the action of the positive modes:

$$
P|p\rangle = p|p\rangle, \quad a_n|p\rangle = 0, \quad \xi_m|p\rangle = 0 \quad n, m > 0. \tag{17}
$$

In the case of the R sector the highest weight becomes doubly degenerate due to the presence of zero mode ξ_0. So, there are two ground states $|p,+\rangle$ and $|p,-\rangle$: $|p,+\rangle = \xi_0|p,-\rangle$. Using the above free field representation of the superconformal algebra one can obtain that for generic $\hat{c}$ and p, F_p is isomorphic to the super-Virasoro module with the highest weight vector $|p\rangle$:

$$
L_0|p\rangle = \Delta_{NS}|p\rangle, \quad \Delta_{NS} = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}-1}{16} \tag{18}
$$

in the NS sector and module with two highest weight vectors in the Ramond case:

$$L_0|p,\pm\rangle = \Delta_R|p,\pm\rangle, \quad \Delta_R = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}}{16}, \quad |p,+\rangle = \frac{\beta^2}{\sqrt{2}p}G_0|p,-\rangle. \tag{19}$$

The space F_p, now considered as super-Virasoro module, splits in the sum of finite-dimensional subspaces, determined by the value of L_0: $F_p = \oplus_{k=0}^{\infty} F_p^{(k)}$, $L_0 F_p^{(k)} = (\Delta+k)F_p^{(k)}$. The quantum versions of local integrals of motion should act invariantly on the subspaces $F_p^{(k)}$. Thus, the diagonalization of IM reduces (in a given subspace $F_p^{(k)}$) to the finite purely algebraic problem, which however rapidly become very complex for large k. It should be noted also that in the case of the Ramond sector G_0 does not commute with IM (even classically), so IM mix $|p,+\rangle$ and $|p,-\rangle$.
At the end of this section we introduce another useful notion – vertex operator. We need two types of them: $V_B^{(a)} = \int d\theta\theta : e^{a\Phi} :$ ("bosonic") and $V_F^{(b)} = \int d\theta : e^{b\Phi} :$ ("fermionic"), where $\Phi(u,\theta) = \phi(u) - \theta\xi(u)$ is a superfield. Thus, $V_B^{(a)} =: e^{a\phi} :$, $V_F^{(b)} = -b\xi : e^{b\phi} :$ and normal ordering here means that $: e^{c\phi(u)} := \exp\Big(c\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu}\Big)\exp\Big(ci(Q+Pu)\Big)\exp\Big(-c\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu}\Big)$.

4 Quantum Monodromy Matrix and Fusion Relations

In this section we will construct the quantum versions of monodromy matrices, operators $\mathbf{L}_s$ and $\mathbf{t}_s$.
The classical monodromy matrix is based on the $\widehat{osp}(1|2)$ affine Lie algebra. In the quantum case the underlying algebra is quantum $\widehat{osp}_q(1|2)$ [14] with $q = e^{i\pi\beta^2}$ and generators, corresponding to even root α_0 and odd root α:

$$\begin{aligned}
&[h_\gamma, h_{\gamma'}] = 0 \quad (\gamma,\gamma' = \alpha, d, \alpha_0), \quad [e_\beta, e_{\beta'}] = \delta_{\beta,\beta'}[h_\beta] \quad (\beta,\beta' = \alpha,\alpha_0),\\
&[h_d, e_{\pm\alpha_0}] = \pm e_{\pm\alpha_0}, \quad [h_d, e_{\pm\alpha}] = 0, \quad [h_{\alpha_0}, e_{\pm\alpha_0}] = 2e_{\pm\alpha_0},\\
&[h_{\alpha_0}, e_{\pm\alpha}] = \mp e_{\mp\alpha}, \quad [h_\alpha, e_{\pm\alpha}] = \pm\frac{1}{2}e_{\pm\alpha}, \quad [h_\alpha, e_{\pm\alpha_0}] = \mp e_{\alpha_0},\\
&[[e_{\pm\alpha}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0, \quad [e_{\pm\alpha}, [e_{\pm\alpha}, [e_{\pm\alpha}[e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm\alpha_0}]_q]_q]_q]_q]_q = 0.
\end{aligned} \tag{20}$$

Here $[,]_q$ is the super q-commutator: $[e_a, e_b] = e_a e_b - q^{(a,b)}(-1)^{p(a)p(b)}e_b e_a$ and parity p is defined as follows: $p(h_{\alpha_0}) = 0$, $p(h_\alpha) = 0$, $p(e_{\pm\alpha_0}) = 0$, $p(e_{\pm\alpha}) = 1$. Also, as usual, $[h_\beta] = \frac{q^{h_\beta}-q^{-h_\beta}}{q-q^{-1}}$. The finite dimensional representations $\pi_s^{(q)}(\lambda)$ of $\widehat{osp}_q(1|2)$ can be characterized by integer number s and have the following explicit form:

$$\begin{aligned}
h_{\alpha_0}|j,m\rangle &= 2m|j,m\rangle,\\
e_{\alpha_0}|j,m\rangle &= \lambda\sqrt{[j-m][j+m+1]}|j,m+1\rangle,
\end{aligned} \tag{21}$$

$$\begin{aligned}
e_{-\alpha_0}|j,m\rangle &= \lambda^{-1}\sqrt{[j+m][j-m+1]}|j,m-1\rangle, \\
e_{\alpha}|j,m\rangle &= \sqrt{\lambda}((-1)^{-2j}\sqrt{\alpha(j)[j-m+1]}|j+1/2,m-1/2\rangle \\
&+ \sqrt{\alpha(j-1/2)[j+m]}|j-1/2,m-1/2\rangle), \\
e_{-\alpha}|j,m\rangle &= \sqrt{\lambda}^{-1}(-\sqrt{\alpha(j)[j+m+1]}|j+1/2,m+1/2\rangle \\
&- (-1)^{2j}\sqrt{\alpha(j-1/2)[j-m]}|j-1/2,m+1/2\rangle), \\
h_{\alpha_0} = -2h_{\alpha} \ &, \ h_d = \frac{1}{2}\lambda\frac{d}{d\lambda} + \frac{1}{4}h_{\alpha_0},
\end{aligned}$$

where $j = 0, 1/2, ..., s/2$, $m = -j, -j+1..., j$. The normalization coefficients

$$\alpha(j) = \frac{[j+1][j+1/2][1/4]}{[2j+2][2j+1][1/2]}\left((-1)^{s-2j+1}\frac{[s+3/2]}{[s/2+3/4]} + \frac{[j+3/2]}{[j/2+3/4]}\right) \tag{22}$$

are the solution of the recurrence relation:

$$\alpha(j)\frac{[2j+2]}{[j+1]} + \alpha(j-1/2)\frac{[2j]}{[j]} = 1, \quad \alpha(s/2) = 0. \tag{23}$$

It should be noted that if we take classical limit $q \to 1$ then it is not hard to calculate that $\alpha(s/2-k) = 0$, if $k < s/2$ is a nonegative integer and $\alpha(s/2-k) = 1/2$, if $k < s/2$ is nonegative half-integer. Using this fact one can obtain that this representation in the classical limit appears to be a direct sum of finite dimensional irreducible representations of $\widehat{osp}(1|2)$: $\pi_s^{(1)}(\lambda) = \oplus_{k=0}^{[s/2]}\pi_{s-2k}(\lambda)$. In this sum k runs through integer and half-integer numbers. One can notice that the structure of irreducible finite dimensional representations of $\widehat{osp}_q(1|2)$ is similar to those of $(A_2^{(2)})_q$ [2]. This is the consequence of the coincidence of their Cartan matrices.

After these preparations we are ready to introduce the quantum analogue of $\mathbf{L}_s$ operators:

$$\begin{aligned}
&\mathbf{L}_s^{(q)} = \pi_s^{(q)}(\lambda)(\mathbf{L}^{(q)}) = \\
&\pi_s^{(q)}(\lambda)\left(e^{-i\pi P h_{\alpha_0}}P\exp\left(\int_0^{2\pi}\mathrm{d}u\Big(: e^{2\phi(u)} : e_{\alpha_0} + \xi(u) : e^{-\phi(u)} : e_{\alpha}\Big)\right)\right).
\end{aligned} \tag{24}$$

One can see that the term $e^{-2\phi(u)}2e_{\alpha}^2$ is missing in the P-exponent in comparison with the classical case (11). This is the general result for superalgebras and we will return to this in [17]. Analyzing the singularity properties of the integrands in P-exponent of $\mathbf{L}_s^{(q)}(\lambda)$ one can find that the integrals are convergent for $-\infty < \hat{c} < 0$ and need regularization for a wider region.

Now let's prove that in the classical limit $\mathbf{L}^{(q)}$ will coincide with $\mathbf{L}$. First let's analyse the products of the operators we have in the P-exponent. The product of the two fermion operators can be written in such a way:

$$\xi(u)\xi(u') =: \xi(u)\xi(u') : -i\beta^2\frac{e^{-\kappa\frac{i}{2}(u-u')}}{e^{\frac{i}{2}(u-u')} - e^{-\frac{i}{2}(u-u')}}. \tag{25}$$

where κ is equal to zero in the NS sector and equal to 1 in the R sector Also, for vertex operators we have:

$$: e^{a\phi(u)} :: e^{b\phi(u')} := (e^{\frac{i}{2}(u-u')} - e^{-\frac{i}{2}(u-u')})^{\frac{ab\beta^2}{2}} : e^{a\phi(u)+b\phi(u')} :, \tag{26}$$

We can rewrite these products, extracting the singular parts:

$$\xi(u)\xi(u') = -\frac{i\beta^2}{(iu - iu')} + \sum_{k=1}^{\infty} c_k(u)(iu - iu')^k, \tag{27}$$

$$: e^{a\phi(u)} :: e^{b\phi(u')} : = (iu - iu')^{\frac{ab\beta^2}{2}} (: e^{(a+b)\phi(u)} : + \sum_{k=1}^{\infty} d_k(u)(iu - iu')^k), \tag{28}$$

where $c_k(u)$ and $d_k(u)$ are operator-valued functions of u.

The $\mathbf{L}^{(q)}(\lambda)$ operator can be expressed in the following way:

$$\begin{aligned} \mathbf{L}^{(q)} &= e^{-i\pi P h_{\alpha_0}} \lim_{N\to\infty} \prod_{m=1}^{N} \tau_m^{(q)}, \qquad \tau_m^{(q)} = P \exp \int_{x_{m-1}}^{x_m} \mathrm{d}u K(u), \\ K(u) &\equiv \, : e^{2\phi(u)} : e_{\alpha_0} + \xi(u) : e^{-\phi(u)} : e_\alpha . \end{aligned} \tag{29}$$

Here we have divided the interval $[0, 2\pi]$ into small intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \Delta = 2\pi/N$. Studying the behaviour of the first two iterations when $\beta^2 \to 0$:

$$\tau_m^{(q)} = 1 + \int_{x_{m-1}}^{x_m} \mathrm{d}u K(u) + \int_{x_{m-1}}^{x_m} \mathrm{d}u K(u) \int_{x_{m-1}}^{u} \mathrm{d}u' K(u') + O(\Delta^2). \tag{30}$$

we conclude that the second iteration can give contribution to the first one. To see this let's consider the expression that comes from the second iteration:

$$-\int_{x_{m-1}}^{x_m} \mathrm{d}u \xi(u) \int_{x_{m-1}}^{u} \mathrm{d}u' \xi(u') : e^{-\phi(u)} :: e^{-\phi(u')} : e_\alpha^2. \tag{31}$$

Now, using the above operator products and seeking the terms of order $\Delta^{1+\beta^2}$ (only those can give us the first iteration terms in $\beta^2 \to 0$ limit) one obtains that their contribution is:

$$\begin{aligned} & i\beta^2 \int_{x_{m-1}}^{x_m} \mathrm{d}u \int_{x_{m-1}}^{u} \mathrm{d}u' (iu - iu')^{\frac{\beta^2}{2} - 1} : e^{-2\phi(u)} : e_\alpha^2 \\ &= 2 \int_{x_{m-1}}^{x_m} \mathrm{d}u : e^{-2\phi(u)} : (iu - ix_{m-1})^{\frac{\beta^2}{2}} e_\alpha^2 . \end{aligned} \tag{32}$$

Considering this in the classical limit we recognize the familiar terms from $\mathbf{L}$:

$$\tau_m^{(1)} = 1 + \int_{x_{m-1}}^{x_m} \mathrm{d}u \Big(\xi(u) e^{-\phi(u)} e_\alpha + e^{2\phi(u)} e_{\alpha_0} + e^{-2\phi(u)} 2 e_\alpha^2 \Big) + O(\Delta^2). \tag{33}$$

Collecting all $\tau_m^{(1)}$ one obtains the desired result: $\mathbf{L}^{(1)} = \mathbf{L}$. Using information about representations one can also get: $\mathbf{L}_s^{(1)}(\lambda) = \sum_{k=0}^{[s/2]} \mathbf{L}_{s-2k}(\lambda)$, where k runs over integer numbers.

Using the properties of quantum R-matrix [3] one obtains that $\mathbf{R}\Delta(\mathbf{L}^{(q)}) = \Delta^{op}(\mathbf{L}^{(q)})\mathbf{R}$, where Δ and Δ^{op} are coproduct and opposite coproduct of $\widehat{osp}_q(1|2)$ [14] correspondingly. Factorizing $\Delta(\mathbf{L}^{(q)})$ and $\Delta^{op}(\mathbf{L}^{(q)})$, according to the properties of vertex operators and P-exponent, we get the so called RTT-relation [3],[4]:

$$\begin{aligned}
&\mathbf{R}_{ss'}(\lambda\mu^{-1})\Big(\mathbf{L}_s^{(q)}(\lambda)\otimes\mathbf{I}\Big)\Big(\mathbf{I}\otimes\mathbf{L}_{s'}^{(q)}(\mu)\Big) \\
&= (\mathbf{I}\otimes\mathbf{L}_{s'}^{(q)}(\mu)\Big)\Big(\mathbf{L}_s^{(q)}(\lambda)\otimes\mathbf{I}\Big)\mathbf{R}_{ss'}(\lambda\mu^{-1}),
\end{aligned} \tag{34}$$

where $\mathbf{R}_{ss'}$ is the trigonometric solution of the corresponding Yang-Baxter equation [16] which acts in the space $\pi_s(\lambda)\otimes\pi_{s'}(\mu)$.

Let's define now the "transfer matrices" which are the quantum analogues of the traces of monodromy matrices: $\mathbf{t}_s^{(q)}(\lambda) = str\pi_s(\lambda)(e^{-i\pi ph_{\alpha_0}}\mathbf{L}_s^{(q)})$. According to the RTT-relation one obtains:

$$[\mathbf{t}_s^{(q)}(\lambda), \mathbf{t}_{s'}^{(q)}(\mu)] = 0. \tag{35}$$

Considering the first nontrivial representation (s=1) it is easy to find the expression for $\mathbf{t}_1^{(q)}(\lambda) \equiv \mathbf{t}^{(q)}(\lambda)$: $\mathbf{t}^{(q)}(\lambda) = 1 - 2\cos(2\pi iP) + \sum_{n=1}^{\infty}\lambda^{2n}Q_n$, where Q_n are nonlocal conservation laws, which (with the use of (35)) are mutually commuting: $[Q_n, Q_m] = 0$. Following [1],[2] we expect also that $\mathbf{t}^{(q)}(\lambda)$ generates local IM as in the classical case. Using (35) again one obtains, expanding $\log(\mathbf{t}^{(q)}(\lambda))$: $[Q_n, I_{2k-1}^{(q)}] = 0$, $[I_{2l-1}^{(q)}, I_{2k-1}^{(q)}] = 0$. The first few orders of expansion in λ^2 of $\mathbf{t}_s^{(q)}(\lambda)$ results in the following fusion relation :

$$\mathbf{t}_s^{(q)}(q^{1/4}\lambda)\mathbf{t}_s^{(q)}(q^{-1/4}\lambda) = \mathbf{t}_{s+1}^{(q)}(q^{\frac{1}{2\beta^2}}\lambda)\mathbf{t}_{s-1}^{(q)}(q^{\frac{1}{2\beta^2}}\lambda) + \mathbf{t}_s^{(q)}(\lambda). \tag{36}$$

This result also reminds the fusion relation for $(A_2^{(2)})_q$ case [2].

5 Discussion

Returning to the quantum Miura transformation (14) it should be noted that one can choose another version:

$$-\beta^2 T(u) =: \phi'^2(u) : -(1-\beta^2)\phi''(u) + \frac{1}{2} : \xi\xi'(u) : + \frac{\epsilon\beta^2}{16}. \tag{37}$$

The reason why we introduce this one is the following: we have two candidates to perturb the model without spoiling the conservation laws, the so-called "integrable perturbations" [19]:

$$V_1 = \int d\theta \int_0^{2\pi} du e^{-\Phi}, \quad V_2 = \int d\theta \int_0^{2\pi} du\theta e^{2\Phi}. \tag{38}$$

Operator V_1 is screening for the deformation (14), the dimension of other one is $h_{1,5}(\hat{c}) - 1/2$. But one should be able also to consider V_2 as screening – that is one should use (37). In this case the dimension of other one is $1/2 + h_{1,2}(c)$ where $c = 13 - 6(\beta^2 + 1/\beta^2)$ is the central charge of Virasoro algebra, generated by (37) without fermion term. When one of these operators is screening, another one is chosen as a perturbation, so one can relate the obtained model with $\widehat{osp}(1|2)$ supersymmetric Toda field theory [18] with the corresponding Lagrangian: $\mathcal{L} = D_{u,\theta}\Phi D_{\bar{u},\bar{\theta}}\Phi - e^{-\Phi} - \theta\bar{\theta}e^{2\Phi}$.

Really, when β^2 is rational the obtained model can be treated as superconformal minimal model, perturbed by $h_{1,5}(\hat{c}) - 1/2$ dimensional operator, or as minimal model, perturbed by $1/2 + h_{1,2}(c)$ dimensional operator. We conjecture, that the same one could obtain from the quantum group reduction of the theory with the mentioned Lagrangian.

The solution of the functional system of equations (36) due to the conjecture of [1] determines the whole set of eigenvalues of $\mathbf{t}_s^{(q)}(\lambda)$ in the model. In the case when q is root of unity, this system should become a closed system of equations (quantum group truncation). Also, due to the results of [1] and [2] one can suppose that these functional equations can be transformed to the so-called Thermodynamic Bethe Ansatz equations [7], giving the description of the "massless S-matrix" theory associated with minimal superconformal field theories.

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